

UNIVERSITY OF CALIFORNIA

Los Angeles

***K'*-theory of a Cohen-Macaulay Local Ring with
n-Cluster Tilting Object**

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Viraj Anil Navkal

2013

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ABSTRACT OF THE DISSERTATION

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Viraj Anil Navkal

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2013

Professor Christian Haesemeyer, Chair

In this dissertation we study the K' -theory of a Henselian CM local ring R which is an isolated singularity and has an n -cluster tilting object M . Our main result is a description of the homotopy fiber of the canonical map from $K'(\text{End}_R(M))$ to $K'(R)$. We also develop a technique for decomposing $K'_1(\text{End}_R(M))$. As we demonstrate, these tools can be used to extract surprisingly explicit information about $K'(R)$ for certain choices of R .

Alexander Merkurjev

Michael Gutperle

Paul Balmer

Christian Haesemeyer, Committee Chair

University of California, Los Angeles

2013

To Ayee, Baba, and Nikhil

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ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Christian Haesemeyer. He understood what many advisors would not – that I needed to work on the problems that interested me most, even if they were outside his main area of expertise. I thank him for hundreds of helpful discussions on these problems. This document would not exist without him.

I also want to thank Paul Balmer and Alexander Merkurjev for directing seminars at UCLA that were extraordinarily helpful. Their comments in these seminars gave me insights that I could not easily have gleaned from books and papers. And I thank Daniel Murfet, for many helpful discussions and also for writing a beautiful set of notes that helped me learn critical material in my early years of graduate school. I am also grateful to Michael Wemyss for showing interest in my work and pointing out several clever applications and additions.

My graduate years would have been both less successful and less enjoyable were it not for Tye Lidman, my classmate and roommate for four years. He helped me navigate through the graduate school labyrinth, and I knew I could always turn to him for professional advice. More important, though, his quest to banish personal failure from his life inspired me and all his other friends to be better people. I hope I have the opportunity to live near him again soon.

VITA

EDUCATION

2007 B.A., Mathematics and Physics, University of Pennsylvania

PUBLICATIONS

2012 “ K' -Theory of a Local Ring of Finite Cohen-Macaulay Type,”
submitted, [arXiv:1108.2000](#).

INVITED TALKS

January 2013 “ G -Theory of a Complete Local Ring of Finite CM Type.”
AMS Session on Categories, K-Theory, and Knots, Joint Mathematics Meetings, San Diego, CA.

December 2012 Topology Seminar, University of California, Riverside.

September 2012 “ K' -Theory of a Complete Local Ring of Finite CM Type.”
University of Essen-Duisburg, Essen, Germany.

August 2011 “Bousfield Localization of Spectra.” West Coast Algebraic
Topology Summer School, University of Washington, Seattle.

1 Introduction

1.1 Motivation

The work in this thesis aims to solve the following problem.

Question 1.1. Let R be the ring of germs of functions in a neighborhood of an isolated singularity – for example, $R = \mathbb{C}[[x, y]]/(x^2 - y^3)$. What can one say about the K' -theory of R ?

The original motivation for Question 1.1 was partly geometric. Suppose R is a hypersurface singularity, i.e. $R = S/(f)$ for some regular local ring S and element $f \in S$; then the dg category $\mathbf{MF}(S, f)$ of matrix factorizations over S with potential w is a dg enrichment of the singularity category $\mathbf{D}^{\text{sing}}(R) := \mathbf{D}^{\text{b}}(R)/\mathbf{D}^{\text{perf}}(R)$, which is known to reflect many important properties of the singularity of R (see, e.g., [Orl04], [Orl09]). When $S = k[[x_1, \dots, x_n]]$, the Hochschild homology of $\mathbf{MF}(S, f)$ has been computed explicitly ([Dyc11, Theorem 5.7]):

$$HH_n(\mathbf{MF}(S, f)) = \begin{cases} S/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) & \text{if } n = \dim(R) \\ 0 & \text{otherwise} \end{cases}$$

Since K -theory and Hochschild homology are closely related, one would expect the K -theory of $\mathbf{MF}(S, f)$ to be an interesting invariant as well. And since there is a long exact sequence of K -groups (see 7.8)

$$\dots \longrightarrow K_i(R) \longrightarrow K'_i(R) \longrightarrow K_i(\mathbf{MF}(S, f)) \longrightarrow K_{i-1}(R) \longrightarrow \dots$$

one could hope to study $K_i(\mathbf{MF}(S, f))$ by studying $K'_i(R)$. I achieved only partial success in carrying out this program; Section 7.3 has a sample computation of $K_1(\mathbf{MF}(S, f))$ for a certain class of polynomials f .

As we shall see, though, Question 1.1 is also interesting in its own right. It is closely related to subtle questions in higher K -theory, and it bears a surprising connection to Iyama's n -Auslander-Reiten theory.

A partial answer to Question 1.1 was given by Auslander and Reiten in 1986. They prove the following theorem; see [AR86, §2, Prop. 2.2] for the original statement or [Yos90, 13.7] for the statement in this form.

Theorem 1.2. *Let R be a Henselian CM local ring of finite representation type. Denote by H the free abelian group on the isomorphism classes of indecomposable maximal Cohen-Macaulay R -modules. Then the map $H \rightarrow K'_0(R)$ sending $[M]$ to $[M]$ is surjective, and its kernel is the subgroup*

$$\langle [N] - [E] + [M] \mid \exists \text{ an Auslander-Reiten sequence } 0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0 \rangle.$$

Thus if one knows the Auslander-Reiten sequences in the category of maximal Cohen-Macaulay R -modules, the above theorem allows explicit computation of $K'_0(R)$. Our main theorem, stated below, is a generalization of Theorem 1.2.

Theorem 1.3. *Let (R, \mathfrak{m}) be a Henselian CM local ring, and assume the category $\text{CM}(R)$ of maximal Cohen-Macaulay R -modules has an n -cluster tilting object L . Let I be the set of isomorphism classes of indecomposable summands of L , and let $I_0 = I \setminus \{[R]\}$. Then there is a long exact sequence*

$$\cdots \rightarrow \bigoplus_{[M] \in I_0} K_i(\kappa_M) \rightarrow K'_i(\Lambda) \rightarrow K'_i(R) \rightarrow \bigoplus_{[M] \in I_0} K_{i-1}(\kappa_M) \rightarrow \cdots \quad (1.4)$$

where

$$\Lambda = \text{End}_R(L)^{\text{op}} \quad \text{and} \quad \kappa_M = (\text{End}_R M)^{\text{op}} / \text{rad}((\text{End}_R M)^{\text{op}}).$$

κ_M is always a division ring and, when R/\mathfrak{m} is algebraically closed, $\kappa_M = R/\mathfrak{m}$.

The long exact sequence ends in a presentation

$$\begin{array}{ccccccc} \bigoplus_{[M] \in I_0} K_0(\kappa_M) & \longrightarrow & K'_0(\Lambda) & \longrightarrow & K'_0(R) & \longrightarrow & 0 \\ & & \parallel & & & & \\ & & \mathbb{Z}^I & & & & \\ & & \parallel & & & & \\ & & \mathbb{Z}^{I_0} & & & & \end{array}$$

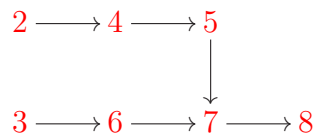
of $K'_0(R)$; when L is an additive generator for $\mathbf{CM}(R)$, it is exactly the one described in Theorem 1.2.

The hypothesis that $\mathbf{CM}(R)$ has an n -cluster tilting object is much weaker than the assumption in Theorem 1.2 that R is of finite representation type; in fact, if R is of finite representation type only if $\mathbf{CM}(R)$ has a 1-cluster tilting object. When R is Gorenstein, $\mathbf{CM}(R)$ has a $(\dim(R) - 1)$ -cluster tilting object if and only if R has a noncommutative crepant resolution.

This document is organized as follows. Sections 2, 3, 4, and 6 give the background we need. Section 2 covers some non-standard facts about additive categories. Section 3 reviews the basic theory of maximal Cohen-Macaulay modules. Section 4 reviews the fundamental theorems in higher K -theory, along with another more obscure theorem (which, in my view, should not be obscure). Section 6 summarizes the main results of n -cluster tilting theory.

Sections 5, 7, and 8 present and develop Theorem 1.3. The goal of Section 5 is to prove Theorem 5.12, which is an important computational tool in later sections but also independently interesting. Section 7 is the heart of the paper; it is there that we prove Theorem 1.3. Section 8 discusses noncommutative localization in K -theory and the difficulty of computing the K -theory of an n -Auslander algebra.

The following diagram depicts the interdependence of the sections. Section x requires results from section y if and only if there is a path from y to x .



1.2 Notation and Conventions

For a ring R , $\text{mod}(R)$ denotes the category of finitely presented left R -modules, $\text{proj}(R)$ denotes the category of finitely generated projective left R -modules, $\text{fl}(R)$ denotes the category of finite length left R -modules, and $\text{rad}(R)$ denotes the Jacobson radical of R . If L is an object of the additive category \mathcal{A} , $\text{add}(L)$ denotes the full subcategory of \mathcal{A} consisting of direct summands of finite direct sums of L . $\text{Ch}^b(\mathcal{A})$ denotes the category of bounded complexes in \mathcal{A} , and $\text{Ch}^b(R) := \text{Ch}^b(\text{mod}(R))$. Since the K -theory functor will appear frequently, we avoid the standard notation $\mathbf{K}^b(\mathcal{A})$ to denote the category of complexes in \mathcal{A} up to homotopy; instead we denote this by $\text{Ho}^b(\mathcal{A})$.

All categories that will arise happen to be additive, and all functors between them are assumed to be additive. All modules over a ring are left modules. All spectra are viewed as objects in the stable homotopy category $\text{Ho}(\text{Sp})$, and maps between spectra are viewed as morphisms in this category. This is useful because $\text{Ho}(\text{Sp})$ is a triangulated category, and the stable homotopy functor $\text{Ho}(\text{Sp}) \rightarrow (\text{abelian groups})$ is a homological functor.

2 Additive Categories

2.1 Coherent Additive Categories

Definition 2.1. Let \mathcal{A} be an additive category. A *(left) \mathcal{A} -module* is an additive functor $\mathcal{A}^{\text{op}} \rightarrow (\text{abelian groups})$. Let $\text{Mod}(\mathcal{A})$ be the category whose objects are \mathcal{A} -modules and whose morphisms are natural transformations. $\text{Mod}(\mathcal{A})$ is an abelian category in which kernels and cokernels are computed objectwise. Call

$F \in \text{Mod}(\mathcal{A})$ *finitely presented* if there is an exact sequence in $\text{Mod}(\mathcal{A})$

$$\mathcal{A}(-, M) \xrightarrow{\mathcal{A}(-, f)} \mathcal{A}(-, M') \longrightarrow F \longrightarrow 0$$

for some $f : M \rightarrow M'$ in \mathcal{A} . Denote by $\text{mod}(\mathcal{A})$ the full subcategory of $\text{Mod}(\mathcal{A})$ consisting of finitely presented functors.

Definition 2.2. Given $f : M' \rightarrow M$ in \mathcal{A} , a *pseudokernel* of f is a map $g : M'' \rightarrow M'$ such that

$$\mathcal{A}(-, M'') \xrightarrow{\mathcal{A}(-, g)} \mathcal{A}(-, M') \xrightarrow{(\mathcal{A})(-, f)} \mathcal{A}(-, M)$$

is exact in $\text{Mod}(\mathcal{A})$. Note that if g is a pseudokernel of f , then g is a kernel of f if and only if $\mathcal{A}(-, g)$ is a monomorphism.

Following [AR86], we call \mathcal{A} *coherent* if every map in \mathcal{A} has a pseudokernel.

Remark 2.3. The definition of coherent above generalizes the classical notion of “coherent” for rings. A ring is usually called *left coherent* if every finitely generated left ideal is finitely presented. A ring R is left coherent if and only if $\text{proj}(R)$ is coherent.

Proposition 2.4 ([Aus66]). *$\text{mod}(\mathcal{A})$ is an abelian category if and only if \mathcal{A} has pseudokernels.*

Definition 2.5. Let \mathcal{B} be a full subcategory of \mathcal{A} and M an object of \mathcal{A} . A *right \mathcal{B} -approximation* of M is a map $f : N \rightarrow M$ in \mathcal{A} with $N \in \mathcal{B}$ such that $\mathcal{B}(-, N) \xrightarrow{\mathcal{A}(-, f)} \mathcal{A}(-, M) |_{\mathcal{B}} \longrightarrow 0$ is exact in $\text{Mod}(\mathcal{B})$. \mathcal{B} is called *contravariantly finite in \mathcal{A}* if every object of \mathcal{A} has a right \mathcal{B} -approximation. Define *left \mathcal{B} -approximation* and *covariantly finite in \mathcal{A}* dually. \mathcal{B} is said to be *functorially finite in \mathcal{A}* if it is both covariantly and contravariantly finite in \mathcal{A} .

Remark 2.6. One sees easily that $\mathcal{B} \subset \mathcal{A}$ is contravariantly finite if and only if for any finitely generated \mathcal{A} -module F , $F |_{\mathcal{B}}$ is a finitely generated \mathcal{B} -module.

Proposition 2.7. *Suppose \mathcal{A} is coherent and $\mathcal{B} \subset \mathcal{A}$ is a contravariantly finite subcategory. Then \mathcal{B} is coherent.*

Proof. Let $f : M \rightarrow N$ be a morphism in \mathcal{B} , and let $i : L \rightarrow M$ be a pseudokernel of f in \mathcal{A} . Let $b : K \rightarrow L$ be a right \mathcal{B} -approximation of L . Then ib is a pseudokernel of f in \mathcal{B} . \square

Remark 2.8. Using Auslander-Buchweitz Theory, we will see in Theorem 3.41 that when R is a Cohen-Macaulay local ring with canonical module, the category $\text{CM}(R)$ of maximal Cohen-Macaulay R -modules is a contravariantly finite subcategory of the coherent additive category $\text{mod}(R)$.

2.2 Krull-Schmidt Additive Categories

Definition 2.9. Let \mathcal{A} be an additive category. An object M of \mathcal{A} is called *indecomposable* if it is not a direct sum of proper submodules. \mathcal{A} is called *Krull-Schmidt* if every object of \mathcal{A} can be decomposed as a finite direct sum of objects with local endomorphism rings.

Note that an object with local endomorphism ring is automatically indecomposable, since a local ring cannot have nontrivial idempotents.

Proposition 2.10. *Let \mathcal{A} be an additive category in which the endomorphism ring of every indecomposable object is local. Suppose $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n$ with M_i, N_j indecomposable in \mathcal{A} for all i, j , then $m = n$ and, for some permutation σ of $\{1, \dots, m\}$, $M_i \cong N_{\sigma(i)}$ for all i .*

Consequently, any object in a Krull-Schmidt category has a unique decomposition into indecomposables.

Definitions 2.11.

1. A commutative local ring (R, \mathfrak{m}) is called *Henselian* if for any monic polynomial $f \in R[t]$, any factorization of f in $(R/\mathfrak{m})[t]$ lifts to a factorization of f in $R[t]$.
2. A ring Λ is called *semiperfect* if it has a complete orthogonal set of idempotents e_1, \dots, e_n such that $e_i R e_i$ is a local ring.

Among commutative local rings, the Henselian ones are those whose module categories are Krull-Schmidt; among all rings, the semiperfect ones are those whose projective module categories are Krull-Schmidt. The next two theorems state these assertions precisely.

Krull-Schmidt Theorem 2.12. *[[Eva73, Theorem 1], [Bon02, Theorem 1.4]]*

A commutative local ring R is Henselian if and only if every module-finite R -algebra which has no nontrivial idempotents is local. In particular, if R is a Henselian local ring, $\text{End}_R(M)$ is local for any indecomposable $M \in \mathbf{mod}(R)$, so $\mathbf{mod}(R)$ is Krull-Schmidt.

Theorem 2.13. *The following conditions on a ring Λ are equivalent.*

1. $\text{proj}(\Lambda)$ is Krull-Schmidt.
2. $\text{proj}(\Lambda^{\text{op}})$ is Krull-Schmidt.
3. $\Lambda/\text{rad}(\Lambda)$ is semisimple and any idempotent in $\Lambda/\text{rad}(\Lambda)$ is the image of an idempotent in Λ .
4. Any simple Λ -module has a projective cover.

Definition 2.14. The *radical* of the additive category \mathcal{A} is the two-sided ideal $\text{rad}_{\mathcal{A}} \subset \mathcal{A}$ defined by

$$\text{rad}_{\mathcal{A}}(X, Y) = \{f \in \text{Hom}_{\mathcal{A}}(X, Y) \mid fg \in \text{rad}(\text{End}_{\mathcal{A}} Y) \text{ for all } g \in \text{Hom}_{\mathcal{A}}(Y, X)\}$$

Proposition 2.15.

1. $\text{rad}_{\mathcal{A}}(X, X)$ is the Jacobson radical $\text{rad}(\text{End}_{\mathcal{A}}(X))$ of $\text{End}_{\mathcal{A}}(X)$.
2. If $X = \bigoplus X_i$, $Y = \bigoplus Y_j$, then $\text{rad}_{\mathcal{A}}(X, Y) = \bigoplus_{i,j} \text{rad}_{\mathcal{A}}(X_i, Y_j)$.
3. If \mathcal{A} is Krull-Schmidt and X and Y are nonisomorphic indecomposables, $\text{rad}_{\mathcal{A}}(X, Y) = \text{Hom}_{\mathcal{A}}(X, Y)$.

From these properties one can see that if \mathcal{A} is Krull-Schmidt, $\text{rad}_{\mathcal{A}}(X, Y)$ consists of those maps $f : X \rightarrow Y$ which do not induce an isomorphism between any indecomposable summands of X and Y .

Let \mathcal{A} be a Krull-Schmidt additive category. We next classify the simple objects of $\text{Mod}(\mathcal{A})$. For each indecomposable object M of \mathcal{A} , let $R_M = \text{End}_R(M)^{\text{op}}$. Let κ_M be the quotient of R_M by its Jacobson radical. R_M is local, so κ_M is a division ring.

Since $\text{End}_R M$ is a local ring, the functor $\mathcal{A}(-, M) \in \text{mod}(\mathcal{A})$ has a unique maximal subfunctor F_M , which coincides with $\text{rad}_{\mathcal{A}}(-, M)$. For any indecomposable $N \not\cong M$, $F_M(N) = \mathcal{A}(N, M)$, and $F_M(M) = \text{rad}(\text{End}_R M)$.

Let $S_M = \mathcal{A}(-, M)/F_M$. As an additive functor, S_M is determined uniquely up to isomorphism by the following properties.

1. $S_M(N) = 0$ if N is indecomposable and not isomorphic to M .
2. $S_M(M) = \kappa_M^{\text{op}}$.
3. For $f : M \rightarrow M$, $S_M(f)(\alpha) = \alpha \cdot \bar{f}$, where \bar{f} is the image of f in κ_M^{op} .

Proposition 2.16. *Let S be a functor in $\text{Mod}(\mathcal{A})$. Then the following are equivalent:*

1. S is simple in $\mathbf{Mod}(\mathcal{A})$.
2. $S \cong S_M$ for some indecomposable $M \in \mathbf{Mod}(\mathcal{A})$.

Proof. $2 \Rightarrow 1$ is clear, since $F_M \subset \mathcal{A}(-, M)$ is a maximal subfunctor. We prove $1 \Rightarrow 2$. Suppose S is a simple object in $\mathbf{Mod}(\mathcal{A})$. S cannot vanish on all indecomposables, so let M be an indecomposable in \mathcal{A} such that $S(M) \neq 0$. Choose nonzero $x \in S(M)$. x defines a nonzero morphism $x_* : \mathcal{A}(-, M) \rightarrow S$ by $(x_*)_N(f) = S(f)(x)$ for $f : N \rightarrow M$ in \mathcal{A} . As S is simple, x_* must be an epimorphism, and its kernel is a maximal subfunctor of $\mathcal{A}(-, M)$. Therefore $S \cong \mathcal{A}(-, M)/F_M = S_M$. \square

3 CM Modules

3.1 Definitions

In this section R denotes a noetherian local ring of Krull dimension d with maximal ideal \mathfrak{m} .

Definition 3.1. Let M be a finitely generated R -module. An n -tuple (x_1, \dots, x_n) of elements of \mathfrak{m} is called an M -sequence of length n if x_{i+1} is a nonzerodivisor on $M/(x_1, \dots, x_i)M$ for $1 \leq i < n$. The *depth* of a finitely generated R -module M is the maximal length of an M -sequence.

Lemma 3.2. For nonzero $M \in \mathbf{mod}(R)$, the following are equivalent:

1. There is an M -sequence of length n .
2. $\mathrm{Ext}_R^i(R/\mathfrak{m}, M) = 0$ for $i < n$.

Proof. ($1 \Rightarrow 2$) Let (x_1, \dots, x_n) be an M -sequence. By induction we may assume $\mathrm{Ext}_R^i(R/\mathfrak{m}, M/x_1M) = 0$ for $i < n - 1$ (if $n = 1$, this condition is of course

vacuous). Then the short exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

induces an exact sequence

$$\cdots \longrightarrow \text{Ext}_R^{i-1}(R/\mathfrak{m}, M/x_1M) \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, M) \xrightarrow{x_1} \text{Ext}_R^i(R/\mathfrak{m}, M) \longrightarrow \cdots$$

As x_1 annihilates $\text{Ext}_R^i(R/\mathfrak{m}, M)$, we conclude by the inductive hypothesis that $\text{Ext}_R^i(R/\mathfrak{m}, M) = 0$ for $i < n$.

(2 \Rightarrow 1) We first prove the following statement: if $\text{Hom}_R(R/\mathfrak{m}, M) = 0$ then there is $r \in \mathfrak{m}$ which is a nonzerodivisor on M . Suppose not; then every element of \mathfrak{m} is a zero divisor on M , so contained in an associated prime of M . By prime avoidance, \mathfrak{m} must be contained in an associated prime of M , so $\mathfrak{m} = \text{ann}(x)$ for some $x \in \mathfrak{m}$. But x induces a nonzero map $R/\mathfrak{m} \rightarrow M$, $\bar{r} \mapsto rx$, which contradicts the assumption $\text{Hom}_R(R/\mathfrak{m}, M) = 0$. The proof now proceeds by induction on n . The case $n = 1$ is exactly the statement we just proved. Let $n > 1$, and assume that $\text{Ext}_R^i(R/\mathfrak{m}, M) = 0$ for $i < n$; we must show there is an M -sequence of length n . Since $\text{Hom}_R(R/\mathfrak{m}, M) = 0$, there is $r \in \mathfrak{m}$ which is a nonzerodivisor on M . The exact sequence

$$0 \longrightarrow M \xrightarrow{r} M \longrightarrow M/rM \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, M) \longrightarrow \text{Ext}_R^i(R/\mathfrak{m}, M/rM) \longrightarrow \text{Ext}_R^{i+1}(R/\mathfrak{m}, M) \longrightarrow \cdots$$

from which we conclude $\text{Ext}_R^i(R/\mathfrak{m}, M/rM) = 0$ for $i < n - 1$. By Nakayama's Lemma $M/rM \neq 0$, so by induction we may assume there is an M/rM -sequence (x_1, \dots, x_{n-1}) of length $n - 1$. (r, x_1, \dots, x_{n-1}) is then an M -sequence of length n . □

Corollary 3.3. For $M \in \text{mod}(R)$,

$$\text{depth}(M) = \min \{i \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \neq 0\}.$$

Since $\text{depth}(M) \leq d$ (see e.g. [Mat89, Theorem 17.2]), we obtain the following.

Corollary 3.4. Let M be a finitely generated R -module. The following are equivalent.

1. $\text{Ext}_R^i(R/\mathfrak{m}, M) = 0$ for $i < d$.
2. $\text{depth}(M) = d$.

Definition 3.5. A finitely generated R -module M is called *maximal Cohen-Macaulay* if it satisfies the equivalent conditions of Corollary 3.4. We denote by $\text{CM}(R)$ the full subcategory of $\text{mod}(R)$ consisting of the maximal Cohen-Macaulay modules. R is called *Cohen-Macaulay* or *CM* if it is maximal Cohen-Macaulay as a module over itself.

Remark 3.6. Suppose R is Cohen-Macaulay. Since R is local, projective R -modules are free; since R is Cohen-Macaulay, free R -modules are maximal Cohen-Macaulay. It is well-known that any maximal Cohen-Macaulay module M is reflexive, i.e. the natural map $M \rightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)$ is an isomorphism. Let $\text{ref}(R) \subset \text{mod}(R)$ be the subcategory consisting of reflexive modules. The previous sentences may be summarized as

$$\text{proj}(R) \subset \text{CM}(R) \subset \text{ref}(R).$$

When R is regular local, the Auslander-Buchsbaum formula

$$\text{proj. dim}(M) + \text{depth}(M) = d$$

holds for any finitely generated R -module M , so in this case $\mathbf{CM}(R) = \mathbf{proj}(R)$. We will sometimes be motivated by the idea that the difference between $\mathbf{CM}(R)$ and $\mathbf{proj}(R)$ should somehow measure the failure of R to be regular local.

Definition 3.7. Motivated by the previous remark, for a CM local ring R we define

$$\underline{\mathbf{CM}}(R) := \mathbf{CM}(R)/\mathbf{proj}(R)$$

Explicitly, $\underline{\mathbf{CM}}(R)$ has the same objects as $\mathbf{CM}(R)$, and $\underline{\mathbf{CM}}(R)(M, N)$ is the quotient of $\mathrm{Hom}_R(M, N)$ by the subgroup of morphisms factoring through a finitely generated projective R -module. We often denote $\underline{\mathbf{CM}}(R)(M, N)$ by $\underline{\mathrm{Hom}}_R(M, N)$.

Remark 3.8. When R is Gorenstein, $\mathbf{CM}(R)$ is a Frobenius category ([Buc87, 4.8]). We do not discuss Frobenius categories until 4.3, but we note now that by the general theory of Frobenius categories, $\underline{\mathbf{CM}}(R)$ is a triangulated category when R is Gorenstein, and the shift functor in $\underline{\mathbf{CM}}(R)$ is the syzygy functor.

The following lemma is straightforward to prove.

Lemma 3.9. *Suppose*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of R -modules.

1. *If M' and M'' are in $\mathbf{CM}(R)$, then so is M .*
2. *If M and M'' are in $\mathbf{CM}(R)$, then so is M' .*

Since $\mathbf{CM}(R)$ is closed under summands in $\mathbf{mod}(R)$, Theorem 2.12 implies $\mathbf{CM}(R)$ is Krull-Schmidt when R is Henselian.

3.2 Isolated Singularities and Auslander-Reiten Theory

In this section, assume R is a Henselian local CM ring of dimension d with canonical module ω .

Definition 3.10. We say $M \in \text{mod}(R)$ is *locally free on the punctured spectrum* if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module for any nonmaximal prime \mathfrak{p} . R is called an *isolated singularity* if $R_{\mathfrak{p}}$ is a regular local ring for every nonmaximal prime \mathfrak{p} .

Lemma 3.11. [[Yos90](#), Lemma 3.3] *The following are equivalent.*

1. R is an isolated singularity.
2. For any $M, N \in \text{CM}(R)$, $\text{Ext}_R^1(M, N)$ is a finite length R -module.
3. Any $M \in \text{CM}(R)$ is locally free on the punctured spectrum.

Definition 3.12. A map $f : E \rightarrow M$ in $\text{CM}(R)$ is called *right almost split* if $\text{im}(\text{CM}(R)(-, f)) = \text{rad}_{\text{CM}(R)}(-, M)$. If M is indecomposable and not isomorphic to R , this is equivalent to saying that

$$(-, E) \xrightarrow{(-, f)} (-, M) \longrightarrow (-, M)/(\text{rad}(-, M)) \longrightarrow 0$$

is exact in $\text{Mod}(\text{CM}(R))$. f is called *minimal right almost split* if f is right almost split and, for any $g \in \text{End}_R E$ such that $fg = f$, g is an automorphism. A short exact sequence

$$0 \longrightarrow N \xrightarrow{g} E \xrightarrow{f} M \longrightarrow 0 \tag{3.13}$$

is called a *(1-)Auslander-Reiten sequence* (or just *AR sequence*) *ending in M* if f is minimal right almost split.

(3.13) is an AR sequence if and only if

$$0 \longrightarrow (-, N) \xrightarrow{(-, g)} (-, E) \xrightarrow{(-, f)} (-, M)$$

is a minimal projective resolution of $(-, M)/\text{rad}(-, M)$ in $\text{Mod}(\text{CM}(R))$. Therefore an AR sequence is determined, up to isomorphism, by its last term.

Remark 3.14. We shall see in Proposition 3.20 that when R is an isolated singularity, any nonfree indecomposable $M \in \text{CM}(R)$ is the last term of an AR sequence.

Definition 3.15. Let $(-)^* = \text{Hom}(-, R) : \text{CM}(R)^{\text{op}} \rightarrow \text{CM}(R)$. For $M \in \text{CM}(R)$, define the functor $\text{Tr} : \underline{\text{CM}}(R)^{\text{op}} \rightarrow \underline{\text{CM}}(R)$, called the *Auslander transpose*, as follows. Given a maximal Cohen-Macaulay module M , choose a presentation

$$P_1 \xrightarrow{d} P_0 \longrightarrow M \longrightarrow 0$$

of M by finitely generated projective modules; then set $\text{Tr } M = \text{coker}(d^*)$. In other words, $\text{Tr } M$ fits into an exact sequence

$$P_0^* \xrightarrow{d^*} P_1^* \longrightarrow \text{Tr } M \longrightarrow 0.$$

Tr is defined in the obvious way on morphisms, and is determined up to canonical natural isomorphism by the choices of projective presentations.

Definition 3.16. Let $\Omega : \underline{\text{CM}}(R) \rightarrow \underline{\text{CM}}(R)$ be the syzygy functor, so ΩM fits into an exact sequence

$$0 \longrightarrow \Omega M \longrightarrow P \longrightarrow M \longrightarrow 0$$

with $P \in \text{proj}(R)$. Set $\tau(M) = \text{Hom}(\Omega^d \text{Tr } M, \omega)$. τ defines a functor from $\underline{\text{CM}}(R)$ to $\overline{\text{CM}}(R) := \text{CM}(R)/\text{add}(\omega)$.

Theorem 3.17 (Auslander-Reiten Duality). *Let $E(R/\mathfrak{m})$ be an injective envelope of R/\mathfrak{m} , and let $D(-) = \text{Hom}_R(-, E(R/\mathfrak{m}))$. Suppose $M \in \text{mod}(R)$ is locally*

free on the punctured spectrum, and suppose $N \in \mathbf{CM}(R)$. Then there is an isomorphism, natural in M and N ,

$$D\underline{\mathbf{Hom}}_R(M, N) \cong \mathrm{Ext}_R^1(M, \tau N) \quad (3.18)$$

Now, suppose M is indecomposable and locally free on the punctured spectrum. $\underline{\mathbf{Hom}}_R(M, M)$ is a local ring, since it is a quotient of the local ring $\mathrm{End}_R(M)$. Therefore $D\underline{\mathbf{Hom}}_R(M, M)$ has a one dimensional socle, consisting of those maps $\underline{\mathbf{Hom}}_R(M, M) \rightarrow E(R/\mathfrak{m})$ which vanish on the Jacobson radical $\mathrm{rad}(\underline{\mathbf{Hom}}_R(M, M))$. Let ξ be a nonzero element of $\mathrm{Ext}_R^1(M, \tau M)$ whose image under the isomorphism (3.18) is in the socle. Then ξ defines a short exact sequence

$$\xi : \quad 0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0. \quad (3.19)$$

Proposition 3.20. *For any indecomposable M in $\mathbf{CM}(R)$ which is not isomorphic to R and locally free on the punctured spectrum, and for any choice of ξ as above, the associated exact sequence (3.19) is an AR sequence. In particular, if R is an isolated singularity, then for every non-free indecomposable $M \in \mathbf{CM}(R)$, there is an AR sequence ending in M .*

Remark 3.21. Recall from 2.16 that the simple objects of $\mathrm{Mod}(\mathbf{CM}(R))$ are the functors $S_M := (-, M)/\mathrm{rad}(-, M)$ for $M \in \mathbf{CM}(R)$ indecomposable. One consequence of Proposition 3.20 is that S_M is finitely presented when M is not isomorphic to R and locally free on the punctured spectrum.

3.3 Finite CM Type

Definition 3.22. We say a CM local ring R is of *finite Cohen-Macaulay type* if there are only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay R -modules.

There are remarkable classification theorems for the rings of finite Cohen-Macaulay type. The classification for hypersurfaces is stated in Theorem 3.24.

Definition 3.23. We say $g \in k[x, y]$ is of *type ADE* if g is one of the following polynomials.

$$(A_n) : x^2 + y^{n+1} \quad (n \geq 1)$$

$$(D_n) : x^2y + y^{n-1} \quad (n \geq 4)$$

$$(E_6) : x^3 + y^4$$

$$(E_7) : x^3 + xy^3$$

$$(E_8) : x^3 + y^5$$

Theorem 3.24. Let k be an algebraically closed field of characteristic not equal to 2, 3, or 5. Suppose $R = k[[x, y, x_2, \dots, x_d]]/(f)$, where $0 \neq f \in (x, y, x_2^2, \dots, x_d^2)$. Then R is of finite CM type if and only if $R \cong k[[x, y, x_2, \dots, x_d]]/(g + x_2^2 + \dots + x_d^2)$ where $g \in k[x, y]$ is of type ADE.

Definition 3.25. We call R an *ADE singularity* if it is of the form described in Theorem 3.24.

The method of proof of Theorem 3.24 is a bit surprising. One direction of the proof – that hypersurfaces of finite CM type are ADE singularities – was proved by Buchweitz-Greuel-Schreyer in 1987 ([BGS87]). On the other hand, the converse was proved separately in dimensions one ([GK85]) and two ([Aus86b], [Esn85]),

using quite different techniques, and then generalized to arbitrary dimensions using a technique called Knörrer periodicity, which we know describe.

For the next definition and theorem, let (S, \mathfrak{n}) be a complete regular local ring and $R = S/(f)$ with $0 \neq f \in \mathfrak{n}^2$. Assume S/\mathfrak{n} is algebraically closed and its characteristic is not 2.

Definition 3.26. Set

$$R^\# := S/(f + z^2).$$

($R^\#$ appears to depend on the choice of S and f , but it turns out to be independent of S and f when S/\mathfrak{n} is algebraically closed of characteristic not equal to 2.)

Let $M \in \mathbf{CM}(R)$. Viewing M as an $R^\#$ -module via the projection $R^\# \rightarrow R$, set

$$M^\# := \mathrm{syz}_1^{R^\#}(M) \in \mathbf{CM}(R^\#).$$

Theorem 3.27 (Knörrer Periodicity). *[[Kno87]] The assignment $M \mapsto M^{\#\#}$ defines a one-to-one correspondence between indecomposable maximal Cohen-Macaulay R -modules and indecomposable maximal Cohen-Macaulay $R^{\#\#}$ -modules. This correspondence extends to an equivalence between stable categories $\underline{\mathbf{CM}}(R) \simeq \underline{\mathbf{CM}}(R^{\#\#})$.*

Remark 3.28. If the equivalence $\underline{\mathbf{CM}}(R) \simeq \underline{\mathbf{CM}}(R^{\#\#})$ can be lifted to an exact functor between exact categories, there should be an isomorphism in K -theory between matrix factorization categories. Unfortunately, I do not see a way to lift the functor.

The converse to Theorem 3.24 now follows from the classification in dimensions one and two (i.e., Theorem 3.24 for $d = 1$ and $d = 2$) together with Theorem 3.27.

3.4 Invariant Theory

In this section we discuss a certain class of singularities, the rational singularities, that is at the same time broad and well-behaved. Rational singularities seem to be much more common than those of finite CM type, but rational singularities still have many of the nice properties of finite CM type singularities.

Let k be a field. Suppose a group G acts on a k -algebra A by algebra endomorphisms. Denote by A^G the invariant subring $\{s \in A \mid (\forall g \in G)(g \cdot s = s)\}$ of A .

Now set $S = k[[x_1, \dots, x_n]]$ and let G be a finite subgroup of $GL_n(k)$ such that $|G|$ is invertible in k . The natural action of G on the subspace $\text{span}\{x_1, \dots, x_n\} \subset S$ extends uniquely to an action on S by k -algebra endomorphisms commuting with infinite sums. We call this action the *linear action of G on S* ; it depends not only on S and G but also on the embedding of G in $GL_n(k)$.

Definition 3.29. Define the *skew group ring* $S * G$ as follows. As an abelian group, $S * G$ is $\bigoplus_{g \in G} Sg$, a direct sum of copies of S . The multiplication is defined by $(sg)(s'g') = (s(g \cdot s'))(gg')$.

An $S * G$ -module is naturally an S -module via restriction of scalars along the inclusion $S = Se \subset S * G$. For $S * G$ -modules M, N , $\text{Hom}_S(M, N)$ has a left G -action (actually, is a left $S * G$ -module) via $(g \cdot f)(m) = g(f(g^{-1}m))$. An S -linear map between $S * G$ -modules commutes with this action if and only if it is $S * G$ -linear; in other words,

$$(\text{Hom}_S(M, N))^G = \text{Hom}_{S * G}(M, N).$$

Since $|G|$ is invertible in k , taking G -invariants $(-)^G$ is exact. Let M and N be $S * G$ -modules and P_\bullet an $S * G$ -free resolution of M . Then P_\bullet is also an S -free

resolution of M , so

$$\begin{aligned}
\mathrm{Ext}_{S * G}^i(M, N) &= H^i \mathrm{Hom}_{S * G}(P_\bullet, N) \\
&= H^i(\mathrm{Hom}_S(P_\bullet, N)^G) \\
&= H^i(\mathrm{Hom}_S(P_\bullet, N))^G \\
&= (\mathrm{Ext}_S^i(M, N))^G.
\end{aligned}$$

Therefore if M is S -projective, $\mathrm{Ext}_{S * G}^i(M, -) \subset \mathrm{Ext}_S^i(M, -) = 0$, so M is $S * G$ -projective. This proves the following.

Proposition 3.30. *$S * G$ has global dimension n .*

Set $R = S^G$. Note that there is a natural ring homomorphism $f : S * G \rightarrow \mathrm{End}_R(S)$ defined by $f(sg)(s') = s(g \cdot s')$; the multiplication in $S * G$ is defined precisely so as to make f a ring homomorphism.

Proposition 3.31. *R is Cohen-Macaulay, and S is a maximal Cohen-Macaulay R -module.*

Definition 3.32. $g \in GL_n(k)$ is called a *pseudo-reflection* if it fixes a codimension-one subspace of k^n .

Theorem 3.33 ([Aus62]). *Suppose G contains no nontrivial pseudo-reflections. Then f is an isomorphism.*

It follows that when G contains no nontrivial pseudo-reflections, there is an equivalence $\mathrm{add}_R(S) \rightarrow \mathrm{proj}(S * G)$ sending M to $\mathrm{Hom}_R(S, M)$, which is a left module over $\mathrm{End}_R(S) = \mathrm{proj}(S * G)$.

Definition 3.34. Let $\mathfrak{n} \subset S * G$ be the Jacobson radical of $S * G$; concretely, \mathfrak{n} is the two-sided ideal generated by x_1, \dots, x_n . Given $V \in \mathrm{mod}(kG)$ and $M \in$

$\text{mod}(S)$, the vector space $M \otimes_k V$ is naturally an $S * G$ -module, with multiplication defined by $(sg)(m \otimes v) = (sg(m)) \otimes (g(v))$. Define functors

$$\begin{aligned} F : \text{mod}(kG) &\longrightarrow \text{proj}(S * G) \\ V &\longmapsto S \otimes_k V \\ H : \text{proj}(S * G) &\longrightarrow \text{mod}(kG) \\ P &\longmapsto P/\mathfrak{n}P \end{aligned}$$

Proposition 3.35. *F and H are inverses on objects; that is, $H(F(V)) \cong V$ and $F(H(P)) \cong P$. In particular, F and H induce a bijection between the set of isomorphism classes of irreducible kG -modules and the set of isomorphism classes of indecomposable $S * G$ -modules.*

Proof. We have

$$\begin{aligned} H(F(V)) &= (S \otimes_k V)/\mathfrak{n}(S \otimes_k V) \\ &= (S \otimes_k V)/(\mathfrak{n}S \otimes_k V) \\ &\cong V. \end{aligned}$$

On the other hand, $(S * G)/\mathfrak{n}$ is semisimple, so $S * G$ is a semiperfect ring, and therefore the projection $p : P \rightarrow P/\mathfrak{n}P$ and $q : S \otimes_k P/\mathfrak{n}P \rightarrow (S \otimes_k P/\mathfrak{n}P)/\mathfrak{n}(S \otimes_k P/\mathfrak{n}P)$ are projective covers. Since $(S \otimes_k P/\mathfrak{n}P)/\mathfrak{n}(S \otimes_k P/\mathfrak{n}P) = (S \otimes_k P/\mathfrak{n}P)/((\mathfrak{n}S) \otimes_k P/\mathfrak{n}P) \cong P/\mathfrak{n}P$, the projective covers must be isomorphic.

□

Remark 3.36. Actually, Proposition 3.35 is just an instance of the following more general facts about any semiperfect ring Λ :

1. The functor $H = (\Lambda/\text{rad}(\Lambda) \otimes_{\Lambda} -) : \text{proj}(\Lambda) \rightarrow \text{mod}(\Lambda/\text{rad}(\Lambda))$ induces a bijection between isomorphism classes of indecomposables. (In fact, H is the quotient of $\text{proj}(\Lambda)$ by its Jacobson radical $\text{rad}_{\text{proj}(\Lambda)} \cdot$)

2. Suppose the quotient $\Lambda \rightarrow \Lambda/\text{rad}(\Lambda)$ has a section $f : \Lambda/\text{rad}(\Lambda) \rightarrow \Lambda$, and let $f^* = (\Lambda \otimes_{\Lambda/\text{rad}(\Lambda)} -) : \text{mod}(\Lambda/\text{rad}(\Lambda)) \rightarrow \text{proj}(\Lambda)$. Then f^* and H induce inverse functions between isomorphism classes of indecomposables.

Theorem 3.37 ([Aus86b]). *Suppose k is algebraically closed, G contains no non-trivial pseudo-reflections, and $n = 2$. Then $\text{add}_R(S) = \text{CM}(R)$. In particular, R is of finite Cohen-Macaulay type.*

3.5 Auslander-Buchweitz Approximation

Let R be a Cohen-Macaulay local ring with dualizing module ω . The next proposition states that the subcategory of $\text{mod}(R)$ consisting of modules of finite injective dimension is “orthogonal” to the subcategory of maximal Cohen-Macaulay modules.

Proposition 3.38 ([AB89]). *Let $M, Z \in \text{mod}(R)$.*

1. *M is MCM if and only if $\text{Ext}_R^i(M, Y) = 0$ for all $i > 0$ and $Y \in \text{mod}(R)$ of finite injective dimension.*
2. *Z is of finite injective dimension if and only if $\text{Ext}_R^i(N, Z) = 0$ for all $i > 0$ and $N \in \text{CM}(R)$.*

Definition 3.39. Let $p : M \rightarrow X$ be a map in $\text{mod}(R)$. p is called *right minimal* if for any $f : M \rightarrow M$ satisfying $pf = p$, f is an isomorphism. p is called an *MCM approximation* of X if M is maximal Cohen-Macaulay and $\ker(p)$ is of finite injective dimension. p is called a *minimal MCM approximation* of X if p is both right minimal and an MCM approximation of X .

A map $i : X \rightarrow Z$ is called an *FID hull* of X if Z is of finite injective dimension and $\text{coker}(i)$ is maximal Cohen-Macaulay.

Lemma 3.40. *Let $p : M \rightarrow X$ be an MCM approximation of X . The following are equivalent:*

1. p is right minimal.
2. p has no nonzero summand of the form $N \rightarrow 0$.

Theorem 3.41 ([AB89]). *Any $M \in \text{mod}(R)$ has both a minimal MCM approximation and an FID hull.*

The proof we provide here is from [LW12, Theorem 11.17].

Proof. The proof is by induction on codepth $M := d - \text{depth } M$.

If codepth $M = 0$, M is a minimal MCM approximation of itself. An FID hull of M can be constructed as follows. Denote by M^\vee the Matlis dual $\text{Hom}_R(-, \omega)$. $(-)^{\vee}$ defines an exact functor $\mathbf{CM}(R) \rightarrow \mathbf{CM}(R)$ ([Yos90, Corollary 1.13]). Choose a surjection $p : F \rightarrow M^\vee$, so there is a short exact sequence

$$0 \longrightarrow \ker(p) \longrightarrow F \xrightarrow{p} M^\vee \longrightarrow 0.$$

Since $\text{Hom}_R(-, \omega)$ is exact on $\mathbf{CM}(R)$ and $\ker(p) \in \mathbf{CM}(R)$, there is a short exact sequence

$$0 \longrightarrow (M^\vee)^\vee \xrightarrow{p^\vee} F^\vee \longrightarrow (\ker(p))^\vee \longrightarrow 0.$$

F^\vee is isomorphic to $\omega^{\oplus n}$ for $n = \text{rank}(F)$, and $(\ker(p))^\vee \in \mathbf{CM}(R)$ because $\ker(p) \in \mathbf{CM}(R)$. Since $M \cong (M^\vee)^\vee$, the above sequence gives an FID hull of M .

Now suppose codepth(M) = $n > 0$. Choose a projective cover $p : F \rightarrow M$ of M , so $K := \ker(p)$ has codepth $n - 1$. By induction K has an FID hull

$$0 \longrightarrow K \longrightarrow Z \longrightarrow N \longrightarrow 0.$$

Now let L be the pushout $F \amalg_K Z$, so there is a diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{p} & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & Z & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & N & \equiv & N & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

with exact rows and columns. L is an extension of two maximal Cohen-Macaulay modules, so $L \in \text{CM}(R)$. Therefore the middle row of the diagram is an MCM approximation of M .

Since L is in $\text{CM}(R)$ it has an FID hull

$$0 \longrightarrow L \longrightarrow Z' \longrightarrow N' \longrightarrow 0.$$

Let L' be the pushout $Z' \amalg_L M$, so there is a commutative diagram

$$\begin{array}{ccccccc}
& & & & 0 & & 0 \\
& & & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z & \longrightarrow & Z' & \longrightarrow & L' \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & N' & \equiv & N' \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

Since Z and Z' have finite injective dimension, so does L' . Therefore the last column in the diagram is an FID hull of M . \square

Proposition 3.42.

1. Let $p : M \rightarrow X$ be an MCM approximation of X , and suppose $h : N \rightarrow X$ is another map with N maximal Cohen-Macaulay. Then h factors through p .
2. Suppose $p : M \rightarrow X$ and $p' : M' \rightarrow X$ are MCM approximations with p right minimal. Then there is a decomposition $M' = M \oplus N$ for some maximal Cohen-Macaulay module N such that $p' |_{M} = p$ and $p' |_{N} = 0$. Consequently a minimal MCM approximation is unique up to isomorphism.

Proof.

1. There is an exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}_R(N, \ker(p)) & \longrightarrow & \text{Hom}_R(N, M) & & \\
 & & & & \searrow^{p \circ -} & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & \longleftarrow & & \\
 & & & & \text{Hom}_R(N, X) & \longrightarrow & \text{Ext}_R^1(N, \ker(p))
 \end{array}$$

By Proposition 3.38, $\text{Ext}_R^1(N, \ker(p)) = 0$. Therefore every map $N \rightarrow X$ factors through p , as desired.

2. The proof above shows that each of p, p' factors through the other, so there is a diagram

$$\begin{array}{ccc}
 M & \xrightarrow{p} & X \\
 \downarrow s & & \parallel \\
 M' & \xrightarrow{p'} & X \\
 \downarrow r & & \parallel \\
 M & \xrightarrow{p} & X
 \end{array}$$

Since $p(rs) = p$ and p is right minimal, rs must be an isomorphism. This implies $M' = \text{im}(s) \oplus \ker(r)$; identifying $\text{im}(s)$ with M gives $p' |_{\text{im}(s)} = p$ and $p' |_{\ker(r)} = 0$.

□

Remark 3.43. One may at first guess that if $N \in \text{mod}(R)$ has codepth t , an MCM approximation of N could be obtained by taking the t^{th} syzygy $\text{syzy}_R^t(N)$ of N , which is at least guaranteed to be maximal Cohen-Macaulay. This is not correct. Instead, the MCM approximation is a dual of a sufficiently high syzygy of a dual of N (at least, when N is Cohen-Macaulay, i.e. its depth equals its Krull dimension; the general case is more complicated). More precisely, letting M^\vee denote the Matlis dual $\text{Ext}_R^{\text{codepth}(M)}(M, \omega)$, an MCM approximation of N is $(\text{syzy}_R^t(N^\vee))^\vee$. See [LW12, Proposition 11.15] for details.

Corollary 3.44. $\text{CM}(R)$ is a contravariantly finite subcategory of $\text{mod}(R)$.

4 K -Theory of Additive and Exact Categories

4.1 Exact Categories and K -theory

Definition 4.1. An *exact category* is an essentially small additive category \mathcal{E} together with a class of sequences $M' \twoheadrightarrow M \twoheadrightarrow M''$, called *conflations*, such that there is a fully faithful additive functor f from \mathcal{E} into an abelian category \mathcal{A} satisfying the following two properties:

1. f reflects exactness; that is, $M' \twoheadrightarrow M \twoheadrightarrow M''$ is a conflation in \mathcal{E} if and only if $0 \rightarrow f(M') \rightarrow f(M) \rightarrow f(M'') \rightarrow 0$ is a short exact sequence in \mathcal{A} .
2. \mathcal{E} is closed under extensions in \mathcal{A} ; that is, if $0 \rightarrow f(M') \rightarrow A \rightarrow f(M'') \rightarrow 0$ is exact in \mathcal{A} , then $A \cong f(M)$ for some M in \mathcal{E} .

A map is called an *inflation* (resp., *deflation*) if it occurs as the first (resp., second) arrow of a conflation. We say \mathcal{E}' is an *exact subcategory* of \mathcal{E} if \mathcal{E}' and

\mathcal{E} are exact categories, \mathcal{E}' is a subcategory of \mathcal{E} , the inclusion functor $\mathcal{E}' \rightarrow \mathcal{E}$ reflects exactness, and \mathcal{E}' is closed under extensions in \mathcal{E} .

Note that the definition of an exact category is self-dual, since the opposite of an abelian category is again abelian. There is also an intrinsic definition of an exact category, which one can piece together easily using the following fact: if \mathcal{E} is an exact category, the Yoneda embedding of \mathcal{E} into the category \mathcal{A} of left exact functors from \mathcal{E} to abelian groups satisfies properties 1 and 2 above.

If each \mathcal{E}_i is an exact category, we endow $\bigoplus \mathcal{E}_i$ with the structure of exact category by setting the conflations to be the sequences which are coordinate-wise conflations. This direct sum is then the coproduct with respect to exact functors between exact categories.

Any additive category \mathcal{A} has an exact structure in which the conflations are the direct-sum sequences

$$0 \longrightarrow M' \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} M' \oplus M \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} M \longrightarrow 0.$$

This defines a left adjoint l to the forgetful functor f from exact categories to (essentially small) additive categories. For an exact category \mathcal{E} , set $\mathcal{E}^\oplus := lf\mathcal{E}$ – that is, \mathcal{E}^\oplus is the additive category \mathcal{E} with the direct sum sequences for conflations. ($l\mathcal{A}$ defines a minimal exact structure on \mathcal{A} . Incidentally, if \mathcal{A} is weakly idempotent complete, it also has a maximal exact structure – see [Cri12].)

Examples 4.2. If R is a ring, the category $\mathbf{mod}(R)$ of finitely presented R -modules is an exact category in which the conflations are the short exact sequences. Similarly, the category $\mathbf{proj}(R)$ of finitely generated projective R -modules is an exact category with short exact sequences for conflations. Note that every conflation in $\mathbf{proj}(R)$ is split.

Definition 4.3. Let \mathcal{E} be an exact category. A complex (X^\bullet, d^\bullet) in \mathcal{E} is called *acyclic* if there is a factorization of each differential d^i

$$\begin{array}{ccccc} \dots & \longrightarrow & X^i & \xrightarrow{d^i} & X^{i+1} & \longrightarrow & \dots \\ & & \searrow p^i & & \nearrow s^{i+1} & & \\ & & & Z^{i+1} & & & \end{array}$$

such that

$$Z^i \xrightarrow{s^i} X^i \xrightarrow{p^i} Z^{i+1}$$

is a conflation for each i . The category $\text{Ac}^b(\mathcal{E})$ is defined to be the full subcategory of $\text{Ch}^b(\mathcal{E})$ consisting of the acyclic complexes. The *bounded derived category* of \mathcal{E} , written $\text{D}^b(\mathcal{E})$, is defined to Verdier quotient of the homotopy category $\text{Ho}^b(\mathcal{E})$ by the triangulated subcategory consisting of those complexes which are homotopy equivalent to acyclic complexes. The theory of derived categories of exact categories is developed in [Kel96a].

A subcategory $w\mathcal{E} \subset \mathcal{E}$ of an exact category \mathcal{E} is called a *subcategory of weak equivalences* if it contains all objects of \mathcal{E} and all isomorphisms in \mathcal{E} and satisfies Waldhausen's Gluing Lemma [Wal83, 1.2]. We define the K -theory of an exact category \mathcal{E} relative to a subcategory of weak equivalences $w\mathcal{E} \subset \mathcal{E}$ using Waldhausen's S -construction [Wal83, 1.3]. It is denoted simply by $K(\mathcal{E})$, and it is an Ω -spectrum whose n th space is

$$K(\mathcal{E})_n = \Omega|wS^n \mathcal{E}|.$$

If no subcategory of weak equivalences is specified, the K -theory of an exact category \mathcal{E} is taken relative to the subcategory $i\mathcal{E}$ of isomorphisms.

Set $K_n(\mathcal{E}) = \pi_n(K(\mathcal{E}))$. For a ring R , set

$$\begin{array}{ll} K(R) := K(\mathbf{proj}(R)) & K_i(R) := \pi_i(K(R)) \\ K'(R) := K(\mathbf{mod}(R)) & K'_i(R) := \pi_i(K'(R)) \end{array}$$

Then $K_0(R)$ is the usual Grothendieck group of R . If R is a commutative local ring, $K_1(R) \cong R^\times$.

4.2 Quillen's Main Theorems

We now recall the main theorems of Quillen K -theory. These theorems will almost always suffice for our purposes. Occasionally, however, we will need the more sophisticated theorems Waldhausen K -theory, and we recall those in Section [4.4](#).

Theorem 4.4 ([Qui73, §3, Cor. 3]). *Suppose*

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

is an exact sequence of exact functors from \mathcal{E} to \mathcal{E}' . Then $\sum_{j=0}^n (-1)^j K(F_j)$ is nulhomotopic, so induces zero maps on K -groups.

Theorem 4.5 ([Qui73, Cor. 1 to Thm. 3]). *Let \mathcal{E} be an exact category and $\mathcal{P} \subset \mathcal{E}$ an exact subcategory. Suppose that*

1. *for any conflation $M \twoheadrightarrow P \twoheadrightarrow P'$ with P and P' in \mathcal{P} , M is isomorphic to an object in \mathcal{P} ; and*
2. *every object of \mathcal{E} has a \mathcal{P} -resolution of finite length.*

Then the inclusion functor induces a homotopy equivalence $K(\mathcal{P}) \simeq K(\mathcal{E})$.

Actually, the hypotheses of Theorem [4.5](#) imply the inclusion $\mathcal{P} \rightarrow \mathcal{E}$ is a derived equivalence. This follows from the following useful fact.

Lemma 4.6 ([Kel96a, Theorem 12.1]). *Let $\mathcal{D} \subset \mathcal{E}$ be an exact subcategory. Suppose that every conflation $B' \twoheadrightarrow B \twoheadrightarrow A$ with A in \mathcal{D} fits into a commutative*

diagram

$$\begin{array}{ccccc}
 A'' & \twoheadrightarrow & A' & \twoheadrightarrow & A \\
 \downarrow & & \downarrow & & \parallel \\
 B' & \twoheadrightarrow & B & \twoheadrightarrow & A
 \end{array}$$

in which A'' and A' are in \mathcal{D} and the top row is a conflation. Then the canonical functor $D^b(\mathcal{D}) \rightarrow D^b(\mathcal{E})$ is fully faithful.

Condition 1 of the resolution theorem 4.5 easily implies that the hypothesis of Lemma 4.6 applies to $\mathcal{P} \subset \mathcal{E}$, and condition 2 then implies the derived functor is essentially surjective.

Definition 4.7. Let \mathcal{X} be an abelian category and $\mathcal{Y} \subset \mathcal{X}$ a Serre subcategory. We say an exact functor $f : \mathcal{X} \rightarrow \mathcal{Z}$ into an abelian category \mathcal{Z} *annihilates* \mathcal{Y} if $f(Y) \cong 0$ for each Y in \mathcal{Y} . A *Serre quotient of \mathcal{X} by \mathcal{Y}* is an exact functor $p : \mathcal{X} \rightarrow \mathcal{W}$ into an abelian category \mathcal{W} such that p annihilates \mathcal{Y} and satisfies the following universality property: given any exact functor $f : \mathcal{X} \rightarrow \mathcal{Z}$ annihilating \mathcal{Y} , there is an exact functor $g : \mathcal{W} \rightarrow \mathcal{Z}$ such that gp is naturally isomorphic to f , and the choice of such g is unique up to natural isomorphism.

Theorem 4.8 ([Qui73, Theorem 5]). *Let \mathcal{A} be an abelian category with a set of isomorphism classes of objects. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory, $i : \mathcal{B} \rightarrow \mathcal{A}$ the inclusion functor, and $p : \mathcal{A} \rightarrow \mathcal{C}$ a Serre quotient of \mathcal{A} by \mathcal{B} . Then*

$$K(\mathcal{B}) \xrightarrow{K(i)} K(\mathcal{A}) \xrightarrow{K(p)} K(\mathcal{C})$$

is a homotopy fiber sequence.

Theorem 4.9 ([Qui73, Thm. 4]). *Let \mathcal{A} be an abelian category and \mathcal{B} a nonempty full subcategory closed under subobjects, quotients, and finite products in \mathcal{A} . Suppose that every object F in \mathcal{A} admits a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

with F_i/F_{i-1} in \mathcal{B} for each i .

Then the inclusion functor induces a homotopy equivalence $K(\mathcal{B}) \simeq K(\mathcal{A})$.

4.3 Frobenius Categories

Definition 4.10. An object X of an exact category \mathcal{E} is called *projective* if the functor $\mathcal{E}(X, -) : \mathcal{E} \rightarrow (\text{abelian groups})$ is exact. We say \mathcal{E} has *enough projectives* if for every object X of \mathcal{E} , there is a deflation $P \twoheadrightarrow X$ with P projective. Define the notions of *injective* and *enough injectives* dually. An exact category is called a *Frobenius category* if it has enough projective and injective objects and the projective and injective objects coincide; we denote the subcategory of projective-injective objects of a Frobenius category \mathcal{E} by $\text{prinj}(\mathcal{E})$. A map of Frobenius pairs is an exact functor taking projective-injective objects to projective-injective objects. The *stable category* $\underline{\mathcal{E}}$ of the Frobenius category \mathcal{E} is the additive quotient of $\mathcal{E}/\text{prinj}(\mathcal{E})$. Define a functor $\Sigma : \underline{\mathcal{E}} \rightarrow \underline{\mathcal{E}}$ on objects by the property that for each X there is a conflation

$$X \twoheadrightarrow I \twoheadrightarrow \Sigma X$$

with $I \in \text{prinj}(\mathcal{E})$, and extend Σ to morphisms in the obvious way. $\underline{\mathcal{E}}$, together with the suspension functor Σ , is a triangulated category in which the exact triangles are exactly those which are isomorphic to the images of conflations.

A *Frobenius pair* \mathcal{E} is a pair $(\mathcal{E}_1, \mathcal{E}_0)$ of Frobenius categories such that \mathcal{E}_0 is a subcategory of \mathcal{E}_1 and the inclusion functor is a map of Frobenius categories. A map of Frobenius pairs $(\mathcal{E}_1, \mathcal{E}_0) \rightarrow (\mathcal{F}_1, \mathcal{F}_0)$ is just a map of Frobenius categories $\mathcal{E}_1 \rightarrow \mathcal{F}_1$ taking \mathcal{E}_0 to \mathcal{F}_0 . The *derived category* $D(\mathcal{E})$ of a Frobenius pair $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_0)$ is defined to be the Verdier quotient $\underline{\mathcal{E}}_1/\underline{\mathcal{E}}_0$.

Example 4.11. If \mathcal{E} is an exact category, $\text{Ch}^b(\mathcal{E})$ is a Frobenius category in

which the projective-injective objects are exactly the contractible complexes, and its stable category is the homotopy category $\mathrm{Ho}^b(\mathcal{E})$. The inclusion $\mathrm{Ac}^b(\mathcal{E}) \rightarrow \mathrm{Ch}^b(\mathcal{E})$ is a map of Frobenius categories, and the derived category of the pair $(\mathrm{Ch}^b(\mathcal{E}), \mathrm{Ac}^b(\mathcal{E}))$ is the bounded derived category of \mathcal{E} .

If $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_0)$ is a Frobenius pair, the subcategory $w_{\mathcal{E}_0}\mathcal{E}_1 \subset \mathcal{E}_1$ consisting of the morphisms in \mathcal{E}_1 that become invertible in $\mathrm{D}(\mathcal{E})$ is a subcategory of weak equivalences. For example, the subcategory $w_{\mathrm{Ac}^b(\mathcal{E})}\mathrm{Ch}^b(\mathcal{E})$ is the subcategory of quasi-isomorphisms in $\mathrm{Ch}^b(\mathcal{E})$. The K -theory $K(\mathcal{E})$ of \mathcal{E} is then defined to be the K -theory of the exact category \mathcal{E}_1 relative to $w_{\mathcal{E}_0}\mathcal{E}_1$.

The next theorem shows that for K -theory, the setting of Frobenius categories subsumes the setting of exact categories.

Theorem 4.12 ([TT90, Theorem 1.11.7]). *Let \mathcal{E} be an exact category and $i\mathcal{E} \subset \mathcal{E}$ the subcategory of isomorphisms in \mathcal{E} . The inclusion of categories with weak equivalences $(\mathcal{E}, i\mathcal{E}) \rightarrow (\mathrm{Ch}^b(\mathcal{E}), w_{\mathrm{Ac}^b(\mathcal{E})}\mathrm{Ch}^b(\mathcal{E}))$, sending an object to the associated complex concentrated in degree zero, induces a homotopy equivalence $K(\mathcal{E}) \simeq K(\mathrm{Ch}^b(\mathcal{E}), \mathrm{Ac}^b(\mathcal{E}))$.*

4.4 Waldhausen Approximation and Localization

For future reference, we state here the cofinality, approximation, and localization theorems for the K -theory of Frobenius categories.

Theorem 4.13 ([Sch06, Proposition 11.15, 11.17]). *Let $f : \mathcal{E} \rightarrow \mathcal{F}$ be a map between Frobenius pairs, and $\mathrm{D}(f) : \mathrm{D}(\mathcal{E}) \rightarrow \mathrm{D}(\mathcal{F})$ the induced functor on derived categories.*

1. (Cofinality) *If $\mathrm{D}(f)$ is a fully faithful inclusion identifying $\mathrm{D}(\mathcal{E})$ with a dense subcategory of $\mathrm{D}(\mathcal{F})$, the homotopy fiber of $K(f) : K(\mathcal{E}) \rightarrow K(\mathcal{F})$ is*

(-1)-connected. Consequently $K_i(f) : K_i(\mathcal{E}) \rightarrow K_i(\mathcal{F})$ is an isomorphism for $i > 0$ and a monomorphism for $i = 0$.

2. *(Approximation) If $D(f)$ is an equivalence, $K(f) : K(\mathcal{E}) \rightarrow K(\mathcal{F})$ is a homotopy equivalence.*

Theorem 4.14. *Let \mathcal{E} be a Frobenius category and $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{E}$ triangulated subcategories. For $i = 0, 1$, let \mathcal{E}_i be the full subcategory of \mathcal{E} consisting of objects in \mathcal{T}_i . Then \mathcal{E}_0 and \mathcal{E}_1 are Frobenius categories, and the inclusion functors induce a homotopy fiber sequence of K -theory spectra*

$$K(\mathcal{E}_1, \mathcal{E}_0) \longrightarrow K(\mathcal{E}, \mathcal{E}_0) \longrightarrow K(\mathcal{E}, \mathcal{E}_1)$$

Remark 4.15. Waldhausen localization 4.14 does not automatically imply Quillen localization 4.8, because not every short exact sequence of abelian categories induces a short exact sequence of bounded derived categories (see [Kel96b, 1.15 Example c])). We emphasize this point because the proof of Corollary 4.18 in the next section will use Quillen localization, and it is not clear how to prove the same statement using Waldhausen localization instead.

4.5 A Homotopy Fiber Sequence

Let \mathcal{A} be a coherent additive category and $\mathcal{B} \subset \mathcal{A}$ a contravariantly finite subcategory. Let $r : \text{mod}(\mathcal{A}) \rightarrow \text{mod}(\mathcal{B})$ be the restriction functor. Let $\text{mod}_0(\mathcal{A}) = \ker(r) \subset \text{mod}(\mathcal{A})$, i.e. $\text{mod}_0(\mathcal{A})$ is the category of finitely presented functors F on \mathcal{A} satisfying $F|_{\mathcal{B}} \cong 0$. $\text{mod}_0(\mathcal{A})$ is a Serre subcategory of $\text{mod}(\mathcal{A})$, and we shall prove in Proposition 4.17 that r is a Serre quotient of $\text{mod}(\mathcal{A})$ by $\text{mod}_0(\mathcal{A})$.

Lemma 4.16. *r has a left adjoint $s : \text{mod}(\mathcal{B}) \rightarrow \text{mod}(\mathcal{A})$. The unit $\eta : \text{id}_{\text{mod}(\mathcal{B})} \rightarrow$*

rs for the adjunction is an isomorphism, and the counit $\varepsilon : sr \rightarrow \text{id}_{\text{mod}(\mathcal{A})}$ becomes invertible upon application of r .

Proof. Define $s : \text{mod}(\mathcal{B}) \rightarrow \text{mod}(\mathcal{A})$ as follows. For each $F \in \text{mod}(\mathcal{B})$, fix a projective presentation $\mathcal{B}(-, B_1) \rightarrow \mathcal{B}(-, B_0) \rightarrow F \rightarrow 0$ of F , and let $s(F) \in \text{mod}(\mathcal{A})$ be the functor fitting into the exact sequence

$$\mathcal{A}(-, B_1) \longrightarrow \mathcal{A}(-, B_0) \longrightarrow s(F) \longrightarrow 0.$$

Given a map $\phi : F \rightarrow F'$ in $\text{mod}(\mathcal{B})$ and chosen projective presentations

$$\mathcal{B}(-, B_1) \rightarrow \mathcal{B}(-, B_0) \rightarrow F \rightarrow 0$$

$$\mathcal{B}(-, B'_1) \rightarrow \mathcal{B}(-, B'_0) \rightarrow F' \rightarrow 0$$

there is a lift $(\phi_0 : B_0 \rightarrow B'_0, \phi_1 : B_1 \rightarrow B'_1)$ of ϕ to the projective presentations.

This lift induces $s(\phi) : s(F) \rightarrow s(F')$ making the following diagram commute.

$$\begin{array}{ccccccc} \mathcal{A}(-, B_1) & \longrightarrow & \mathcal{A}(-, B_0) & \longrightarrow & s(F) & \longrightarrow & 0 \\ \downarrow \mathcal{A}(-, \phi_1) & & \downarrow \mathcal{A}(-, \phi_0) & & \downarrow s(\phi) & & \\ \mathcal{A}(-, B'_1) & \longrightarrow & \mathcal{A}(-, B'_0) & \longrightarrow & s(F') & \longrightarrow & 0 \end{array}$$

The choice (ϕ_0, ϕ_1) is unique up to homotopy: given another choice (ϕ'_0, ϕ'_1) , $\phi_0 - \phi'_0$ factors through B'_1 . In this case $\mathcal{A}(-, \phi_0) - \mathcal{A}(-, \phi'_0)$ factors through $\mathcal{A}(-, B'_1)$, so the two lifts induce the same map $s(\phi)$. Therefore the choice of $s(\phi)$ is independent of the choices of ϕ_i . One easily verifies that this implies that s is functorial.

Define the adjunction map $\sigma : \text{mod}(\mathcal{B})(G, r(F)) \rightarrow \text{mod}(\mathcal{A})(s(G), F)$ as follows. First define σ for $G = \mathcal{B}(-, B)$ and $F = \mathcal{A}(-, A)$ representable: in this case, $s(G)$ is canonically isomorphic to $\mathcal{A}(-, B)$, and we define σ to be the composition of Yoneda isomorphisms $\text{mod}((\mathcal{B})(-, B), \mathcal{A}(-, A) |_{\mathcal{B}}) \cong (\mathcal{A}(-, A) |_{\mathcal{B}})$

$(B) = \mathcal{A}(B, A) \cong \mathbf{mod}(\mathcal{A})(\mathcal{A}(-, B), \mathcal{A}(-, A))$. Since both $\mathbf{mod}(\mathcal{B})(G, r(F))$ and $\mathbf{mod}(\mathcal{A})(s(G), F)$ are right exact in F when G is representable, and since F is a finitely presented functor, our definition of σ extends uniquely to an isomorphism, natural in G and F , defined on all representable G and finitely presented F . Now since both $\mathbf{mod}(\mathcal{B})(G, r(F))$ and $\mathbf{mod}(\mathcal{A})(s(G), F)$ are contravariant left exact in G , σ extends uniquely to all finitely presented G .

The unit $G \rightarrow rs(G)$ is clearly an isomorphism for G representable, so it is an isomorphism for all finitely presented G . It is a formal consequence that $r(\varepsilon : sr \rightarrow \mathrm{id}_{\mathbf{mod}(\mathcal{A})})$ is an isomorphism. \square

Proposition 4.17. $r : \mathbf{mod}(\mathcal{A}) \rightarrow \mathbf{mod}(\mathcal{B})$ is a Serre quotient of $\mathbf{mod}(\mathcal{A})$ by $\mathbf{mod}_0(\mathcal{A})$.

Proof. Let $f : \mathbf{mod}(\mathcal{A}) \rightarrow \mathcal{Z}$ by an exact functor annihilating $\mathbf{mod}_0(\mathcal{A})$; we need to show there is an exact functor $g : \mathbf{mod}(\mathcal{B}) \rightarrow \mathcal{Z}$, unique up to isomorphism, satisfying $gr \cong f$. Set $g = fs$, where s is the left adjoint to r from Lemma 4.16.

We first prove g is exact. Since s is a left adjoint, s is right exact, so g is right exact. To see that g is left exact, suppose $\iota : F \rightarrow G$ is a monomorphism in $\mathbf{mod}(\mathcal{B})$. Since $\mathrm{id} \cong rs$, $rs(\iota)$ is injective, so $\ker(s(\iota)) \in \mathbf{mod}_0(\mathcal{A})$. Since f is exact and annihilates $\mathbf{mod}_0(\mathcal{A})$, $\ker(fs(\iota)) \cong f(\ker(s(\iota))) \cong 0$, so $g(\iota) = fs(\iota)$ is a monomorphism.

Next we show $f \cong gr$. By Lemma 4.16, $r(\varepsilon_F) : rsr(F) \rightarrow r(F)$ is invertible for any $F \in \mathbf{mod}(\mathcal{A})$, so $\ker(\varepsilon_F)$ and $\mathrm{coker}(\varepsilon_F)$ are in $\mathbf{mod}_0(\mathcal{A})$. Since f annihilates $\mathbf{mod}_0(\mathcal{A})$, $f(\varepsilon_F)$ must be an isomorphism. This proves that $f(\varepsilon) : ge = fse \rightarrow f$ is an isomorphism, as desired.

g satisfies the necessary uniqueness property: if $g' : \mathbf{mod}(\mathcal{B}) \rightarrow \mathcal{Z}$ is another exact functor satisfying $g'e \cong f$, then $g' \cong g'es \cong fs \cong ges \cong g$. \square

Corollary 4.18. *Let \mathcal{A} be a coherent additive category, $\mathcal{B} \subset \mathcal{A}$ a contravariantly finite subcategory, and $r : \text{mod}(\mathcal{A}) \rightarrow \text{mod}(\mathcal{B})$ the restriction. There is a homotopy fiber sequence of K -theory spectra*

$$K(\ker(r)) \longrightarrow K(\text{mod}(\mathcal{A})) \longrightarrow K(\text{mod}(\mathcal{B}))$$

Proof. This is a direct consequence of Proposition 4.17 and Theorem 4.8. \square

5 K_1 of a Krull-Schmidt Additive Category

The main result of this section is the exact sequence of Theorem 5.12, which can be used to decompose K_1 of a Krull-Schmidt additive category. Later we will study the homotopy fiber sequence (7.5), whose middle term is the K -theory of a Krull-Schmidt additive category, and the results of this section will be useful then.

5.1 K_1 of an Additive Category

Let \mathcal{A} be an additive category. We shall consider \mathcal{A} to be an exact category with the split exact structure, in which the conflations are the sequences isomorphic to a direct sum sequence $A \rightarrow A \oplus B \rightarrow B$. Set $\text{Aut}(\mathcal{A})$ to be the category whose objects are pairs (A, ϕ) with A in \mathcal{A} and $\phi \in \text{Aut}_{\mathcal{A}} A$, and whose morphisms are defined by

$$\text{Hom}_{\text{Aut}(\mathcal{A})}((A, \phi), (A', \phi')) = \{f \in \text{Hom}_{\mathcal{A}}(A, A') \mid \phi' f = f \phi\}.$$

$\text{Aut}(\mathcal{A})$ has an exact structure in which a sequence is a conflation iff it is a conflation in \mathcal{A} . Recall from ([She82, §3]) that there is a natural surjection

$$K_0(\text{Aut}(\mathcal{A})) \longrightarrow K_1(\mathcal{A})$$

whose kernel is generated by elements of the form $[(A, \alpha\beta)] - [(A, \alpha)] - [(A, \beta)]$. Denote by $[A, \phi]$, or just $[\phi]$, the image in $K_1(\mathcal{A})$ of the K_0 -class of the object (A, ϕ) of $\text{Aut}(\mathcal{A})$.

Lemma 5.1. *Let $f : A \rightarrow A'$ be a morphism in \mathcal{A} , and let $\phi = \begin{pmatrix} 1_A & 0 \\ f & 1_{A'} \end{pmatrix} \in \text{Aut}_{\mathcal{A}}(A \oplus A')$. Then in $K_1(\mathcal{A})$, $[\phi] = 0$. Similarly, given any $g : A' \rightarrow A$, $[\begin{pmatrix} 1_A & g \\ 0 & 1_{A'} \end{pmatrix}] = 0$.*

Proof. There is a conflation in $\text{Aut}(\mathcal{A})$

$$(A', 1_{A'}) \hookrightarrow (A \oplus A', \phi) \twoheadrightarrow (A, 1_A)$$

so that $[\phi] = [1_{A'}] + [1_A] = 0$. The second statement is proved the same way. \square

Remark 5.2. Suppose $\phi, \psi \in \text{Aut}_{\mathcal{A}}(\bigoplus A_i)$ are row- or column-equivalent – that is, the matrices defining ϕ and ψ differ only up to left- or right-multiplication by elementary matrices, which are identity along the diagonal and zero off the diagonal except in one entry. Then using the lemma above, one sees easily that $[\phi] = [\psi]$.

5.2 Automorphisms in a Krull-Schmidt Category

The next lemma gives us an easy criterion for recognizing automorphisms in a Krull-Schmidt additive category.

Lemma 5.3. *Suppose \mathcal{A} is Krull-Schmidt and $A = \bigoplus_{i=1}^d A_i^{n_i}$ with A_1, \dots, A_d pairwise nonisomorphic indecomposables in \mathcal{A} . Suppose $\phi \in \text{End}_{\mathcal{A}}A$, and denote by ϕ_{ij} the induced morphism $A_j^{n_j} \rightarrow A_i^{n_i}$. Then ϕ is invertible if and only if for all i , ϕ_{ii} is invertible.*

Proof. Let $R = \text{End}_{\mathcal{A}}A$ and $R_i = \text{End}_{\mathcal{A}}(A_i^{n_i})$. Using Proposition 2.15, one sees

that the Jacobson radical of R is

$$\text{rad}(R) = \text{rad}_{\mathcal{A}}(\oplus A_i^{n_i}, \oplus A_j^{n_j}) = \{\phi \in R \mid \phi_{ii} \in \text{rad}(R_i) \text{ for all } i\},$$

so that $R/\text{rad}(R) = \prod R_i/\text{rad}(R_i)$. Therefore ϕ is invertible iff $\bar{\phi}$ is invertible in $R/\text{rad}(R)$, iff $\bar{\phi}_{ii}$ is invertible in $R_i/\text{rad}(R_i)$ for all i , iff ϕ_{ii} is invertible for all i . \square

Using this lemma we can deduce the following.

Corollary 5.4. *Assume \mathcal{A} is Krull-Schmidt, and let $\mathcal{B} \subset \mathcal{A}$ be a full additive category closed under summands. Let A be an object of \mathcal{A} which has no nonzero summand in \mathcal{B} , and let B be an object of \mathcal{B} . Let $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Aut}_{\mathcal{A}}(A \oplus B)$. Then a and d are invertible.*

5.3 Localization of an Additive Category

Let \mathcal{A} be an additive category, and let $\mathcal{B} \subset \mathcal{A}$ be a full additive subcategory closed under summands. Let $e : \mathcal{B} \rightarrow \mathcal{A}$ be the inclusion and $s : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ the quotient of \mathcal{A} by the ideal consisting of morphisms factoring through \mathcal{B} . Let $w \subset \mathcal{A}$ be the multiplicative set consisting of maps which are a composition of the form

$$A \xrightarrow{i} A \oplus B \xrightarrow{\cong} A' \oplus B' \xrightarrow{p} A'$$

with A and A' in \mathcal{A} , B and B' in \mathcal{B} , i and p the canonical inclusion and projection, and the middle map an isomorphism. It is easy to check that w is closed under sums and compositions. An additive functor out of \mathcal{A} sends all morphisms in w to isomorphisms iff it sends all objects in \mathcal{B} to zero, so s is initial among additive functors inverting w . In this sense \mathcal{A}/\mathcal{B} is simultaneously the quotient of \mathcal{A} by \mathcal{B} and the localization of \mathcal{A} at w .

Consider the following conditions on a morphism $f : C \rightarrow C'$:

1. For any choice of isomorphisms $\alpha : C \xrightarrow{\sim} A \oplus B$, $\alpha' : C' \xrightarrow{\sim} A' \oplus B'$, with B, B' in \mathcal{B} and A, A' not having any nonzero summands in \mathcal{B} , the composition

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{\alpha' f \alpha^{-1}} A' \oplus B' \xrightarrow{p_{A'}} A' \quad (5.5)$$

is an isomorphism.

2. For some choice of isomorphisms $\alpha : C \xrightarrow{\sim} A \oplus B$, $\alpha' : C' \xrightarrow{\sim} A' \oplus B'$, with B, B' in \mathcal{B} and A, A' not having any nonzero summands in \mathcal{B} , (5.5) is an isomorphism.
3. f is in w .

Lemma 5.6. *Among the above conditions, $1 \Rightarrow 2 \Rightarrow 3$. If \mathcal{A} is Krull-Schmidt, then $3 \Rightarrow 1$.*

Proof. $1 \Rightarrow 2$ is trivial. To show $2 \Rightarrow 3$, say $\alpha' f \alpha^{-1} : A \oplus B \rightarrow A' \oplus B'$ is given by the matrix $\begin{pmatrix} \phi & a \\ b & c \end{pmatrix}$. Then

$$\begin{array}{ccc} A \oplus B & \xrightarrow{\alpha' f \alpha^{-1}} & A' \oplus B' \\ i_{A \oplus B} \downarrow & & \uparrow p_{A' \oplus B'} \\ A \oplus B \oplus B' & \xrightarrow{\begin{pmatrix} \phi & a & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix}} & A' \oplus B \oplus B' \end{array}$$

commutes, and the lower horizontal map is an isomorphism with inverse $\begin{pmatrix} \phi^{-1} & -\phi^{-1}a & 0 \\ 0 & 1 & 0 \\ -b\phi^{-1} & b\phi^{-1}a-c & 1 \end{pmatrix}$. So $\alpha' f \alpha^{-1}$ is in w , and therefore f is in w .

To show $3 \Rightarrow 1$, assume \mathcal{A} is Krull-Schmidt and f is in w . Given decompositions $\alpha : C \xrightarrow{\sim} A \oplus B$, $\alpha' : C' \xrightarrow{\sim} A' \oplus B'$, there is by hypothesis a commuting

diagram as below.

$$\begin{array}{ccc}
A & \longrightarrow & A' \\
i_A \downarrow & & p_{A'} \uparrow \\
A \oplus B & \xrightarrow{\alpha' f \alpha^{-1}} & A' \oplus B' \\
i_{A \oplus B} \downarrow & & p_{A' \oplus B'} \uparrow \\
A \oplus B \oplus B_1 & \xrightarrow{\cong} & A' \oplus B' \oplus B'_1
\end{array}$$

By Corollary 5.4, the top horizontal map is an isomorphism. □

Remark 5.7. If \mathcal{A} is Krull-Schmidt, it follows from the above characterization of maps in w that w satisfies the 2-out-of-3 property: if two out of f , g , and $f \circ g$ are in w then all three are. In this case w is exactly the class of maps in \mathcal{A} which become invertible in \mathcal{A}/\mathcal{B} .

5.4 The Exact Sequence

Now we proceed to our destination, Theorem 5.12. We adopt the notation of the previous section, with the added assumption that \mathcal{A} is Krull-Schmidt. In this case \mathcal{B} and \mathcal{A}/\mathcal{B} are automatically Krull-Schmidt.

Lemma 5.8. *Suppose A and A' are objects of \mathcal{A} with no nonzero summands in \mathcal{B} . Suppose $\psi \in \mathcal{A}(A, A')$, and assume the image $\overline{\psi}$ of ψ in \mathcal{A}/\mathcal{B} is an isomorphism. Then ψ is itself an isomorphism.*

Proof. First we show that

$$\ker(\text{End}_{\mathcal{A}} A \rightarrow \text{End}_{\mathcal{A}/\mathcal{B}} A) \subset \text{rad}(\text{End}_{\mathcal{A}} A).$$

That is, we show that given morphisms $A \xrightarrow{f} B \xrightarrow{g} A$ with B in \mathcal{B} , $gf \in \text{rad}(\text{End}_{\mathcal{A}} A)$. It suffices, using the characterization of $\text{rad}(\text{End}_{\mathcal{A}} A)$ in the proof of Lemma 5.3, to show that for each indecomposable summand $A_i \subset A$, the

induced map $A_i \xrightarrow{f_i} B \xrightarrow{g_i} A_i$ is in $\text{rad}(\text{End}_{\mathcal{A}}A_i)$. But this follows from the fact that $\text{End}_{\mathcal{A}}A_i$ is local and A_i is not a summand of B .

Since $\bar{\psi}$ is invertible, there is $\phi \in \mathcal{A}(A', A)$ such that $\phi\psi - \text{id}_A \in \ker(\text{End}_{\mathcal{A}}A \rightarrow \text{End}_{\mathcal{A}/\mathcal{B}}A)$ and $\psi\phi - \text{id}_{A'} \in \ker(\text{End}_{\mathcal{A}}A' \rightarrow \text{End}_{\mathcal{A}/\mathcal{B}}A')$. Therefore $\phi\psi - \text{id}_A \in \text{rad}(\text{End}_{\mathcal{A}}A)$ and (by the same argument) $\psi\phi - \text{id}_{A'} \in \text{rad}(\text{End}_{\mathcal{A}}A')$, so $\phi\psi$ and $\psi\phi$ are both invertible. It follows that ψ is invertible. \square

Lemma 5.9. *Suppose A is an object of \mathcal{A} with no nonzero summand in \mathcal{B} , B is an object of \mathcal{B} , and $\alpha = \begin{pmatrix} \phi & a \\ b & c \end{pmatrix} \in \text{Aut}_{\mathcal{A}}(A \oplus B)$. Then $[\alpha] - [\phi] \in \text{im } K_1(e)$.*

Proof. Note that ϕ is an automorphism by Corollary 5.4. Using Remark 5.2, compute:

$$\begin{aligned} [A \oplus B, \begin{pmatrix} \phi & a \\ b & c \end{pmatrix}] &= [A \oplus B, \begin{pmatrix} \phi & a \\ 0 & c - b\phi^{-1}a \end{pmatrix}] \\ &= [A \oplus B, \begin{pmatrix} \phi & 0 \\ 0 & c - b\phi^{-1}a \end{pmatrix}] \\ &= [A, \phi] + [B, c - b\phi^{-1}a] \\ &\equiv [A, \phi] \pmod{\text{im } K_1(e)} \end{aligned}$$

\square

Lemma 5.10. *Suppose (A, ϕ) and (A, ψ) are objects of $\text{Aut}(\mathcal{A})$ such that $\bar{\phi} = \bar{\psi}$ in \mathcal{A}/\mathcal{B} . Then $[\phi] - [\psi] \in \text{im } K_1(e)$.*

Proof. Since $\bar{\phi} = \bar{\psi}$, $\phi - \psi$ factors as a composition $A \xrightarrow{f} B \xrightarrow{g} A$ through some object B of \mathcal{B} . Using Remark 5.2, compute:

$$\begin{aligned} [A, \phi] &= [A \oplus B, \begin{pmatrix} \phi & 0 \\ 0 & 1_B \end{pmatrix}] \\ &= [A \oplus B, \begin{pmatrix} \phi & 0 \\ f & 1_B \end{pmatrix}] \\ &= [A \oplus B, \begin{pmatrix} \phi^{-gf} & -g \\ f & 1_B \end{pmatrix}] \\ &\equiv [A, \psi] \pmod{\text{im } K_1(e)}, \text{ by Lemma 5.9} \end{aligned}$$

Therefore $[A, \phi] \equiv [A, \psi] \pmod{\text{im } K_1(e)}$. \square

Lemma 5.11. *Suppose (C, α) and (C', α') are objects of $\text{Aut}(\mathcal{A})$ such that $(C, \bar{\alpha}) \cong (C', \bar{\alpha}')$ in $\text{Aut}(\mathcal{A}/\mathcal{B})$. Then $[\alpha] - [\alpha'] \in \text{im } K_1(e)$.*

Proof. We may assume there are decompositions

$$(C, \alpha) = (A \oplus B, \begin{pmatrix} \phi & a \\ b & c \end{pmatrix}) \quad (C', \alpha') = (A' \oplus B', \begin{pmatrix} \phi' & a' \\ b' & c' \end{pmatrix})$$

such that A and A' have no nonzero summand in \mathcal{B} and B and B' are in \mathcal{B} . By assumption there is $\beta : A \oplus B \rightarrow A' \oplus B'$ such that $\bar{\beta}$ is an isomorphism in \mathcal{A}/\mathcal{B} and $\bar{\beta}\bar{\alpha} = \bar{\alpha}'\bar{\beta}$. Say β is given by a matrix of the form $\begin{pmatrix} \psi & * \\ * & * \end{pmatrix}$. By Lemma 5.8, ψ must be an isomorphism. Now, $\bar{\beta}\bar{\alpha} = \bar{\alpha}'\bar{\beta} \Rightarrow \bar{\psi}\bar{\phi} = \bar{\phi}'\bar{\psi} \Rightarrow \bar{\phi} = \bar{\psi}^{-1}\bar{\phi}'\bar{\psi}$ so

$$\begin{aligned} [C, \alpha] &\equiv [A, \phi] \pmod{\text{im } K_1(e)}, \text{ by Lemma 5.9} \\ &\equiv [A, \psi^{-1}\phi'\psi] \pmod{\text{im } K_1(e)}, \text{ by Lemma 5.10} \\ &= [A', \phi'] \\ &\equiv [C', \alpha'] \pmod{\text{im } K_1(e)}, \text{ by Lemma 5.9} \end{aligned}$$

\square

Theorem 5.12. *Suppose \mathcal{A} is Krull-Schmidt. Then the sequence*

$$\begin{array}{ccccccc} K_1(\mathcal{B}) & \xrightarrow{K_1(e)} & K_1(\mathcal{A}) & \xrightarrow{K_1(s)} & K_1(\mathcal{A}/\mathcal{B}) & & \\ & & & & \searrow & & \\ & & & & 0 & & \\ & & & & \swarrow & & \\ & & & & & & \\ & \xrightarrow{\quad} & K_0(\mathcal{B}) & \xrightarrow{K_0(e)} & K_0(\mathcal{A}) & \xrightarrow{K_0(s)} & K_0(\mathcal{A}/\mathcal{B}) \longrightarrow 0 \end{array}$$

is exact.

Proof. We show only that the sequence is exact at $K_1(\mathcal{A}/\mathcal{B})$ and $K_1(\mathcal{A})$; the rest is easy. To show $K_1(s)$ is surjective, take $[A, \phi] \in K_1(\mathcal{A}/\mathcal{B})$; after replacing (A, ϕ) by an isomorphic object of $\text{Aut}(\mathcal{A}/\mathcal{B})$, we may assume A has no nonzero summands

in \mathcal{B} . Using Lemma 5.8, we see that any lift $\tilde{\phi}$ of ϕ to \mathcal{A} is automatically an automorphism of A . Hence $K_1(s)([\tilde{\phi}]) = [\phi]$, as desired.

Next we show the sequence is exact at $K_1(\mathcal{A})$. Since $K_1(s)$ is surjective, any element of $\ker K_1(s)$ may be written as a sum of elements of the form

1. $[A, \phi] - [A', \phi']$, for some $\phi \in \text{Aut}_{\mathcal{A}}A$ and $\phi' \in \text{Aut}_{\mathcal{A}}A'$ with $(A, \bar{\phi}) \cong (A', \bar{\phi}')$ in $\text{Aut}(\mathcal{A}/\mathcal{B})$
2. $[A, \phi] - [A, \alpha] - [A, \beta]$ for some $\phi, \alpha, \beta \in \text{Aut}_{\mathcal{A}}A$ with $\overline{\alpha\beta} = \bar{\phi}$ in \mathcal{A}/\mathcal{B}
3. $[A, \phi] - [A', \phi'] - [A'', \phi'']$, for some (A, ϕ) , (A', ϕ') , and (A'', ϕ'') in $\text{Aut}(\mathcal{A})$ such that there is a conflation in $\text{Aut}(\mathcal{A}/\mathcal{B})$

$$(A', \bar{\phi}') \twoheadrightarrow (A, \bar{\phi}) \twoheadrightarrow (A'', \bar{\phi}'')$$

We need to check that any such element is in $\text{im } K_1(e)$. Since an element of the first form is also of the third form, we skip the check for elements of the first form. For elements of the second form, observe that

$$[A, \phi] - [A, \alpha] - [A, \beta] = [A, \phi] - [A, \alpha\beta] \in \text{im } K_1(e)$$

by Lemma 5.10. So it remains to show any element of the third form is in $\text{im } K_1(e)$.

Given (A, ϕ) , (A', ϕ') , and (A'', ϕ'') as in 3. above, there is an isomorphism in $\text{Aut}(\mathcal{A}/\mathcal{B})$

$$(A, \bar{\phi}) \cong \left(A' \oplus A'', \begin{pmatrix} \bar{\phi}' & \bar{h} \\ 0 & \bar{\phi}'' \end{pmatrix} \right)$$

for some $h : A'' \rightarrow A'$. Since $\begin{pmatrix} \phi' & h \\ 0 & \phi'' \end{pmatrix}$ is invertible, it follows by Lemma 5.11 that

$$\begin{aligned} [A, \phi] &\equiv \left[A' \oplus A'', \begin{pmatrix} \phi' & h \\ 0 & \phi'' \end{pmatrix} \right] \pmod{\text{im } K_1(e)} \\ &= [A', \phi'] + [A'', \phi''] \end{aligned}$$

and therefore $[A, \phi] - [A', \phi'] - [A'', \phi''] \in \text{im } K_1(e)$. \square

we discussed classical Auslander-Reiten theory in [3.2](#). But we will sometimes need the extra generality when we apply the results from this section.

Definition 6.1. Let T be a complete regular local ring. A T -order is a T -algebra R which is finitely generated projective as a left T -module. Note R may not be commutative.

Let R be a T -order. R is called an *isolated singularity* if

$$\text{gl. dim}(R \otimes_T T_{\mathfrak{p}}) = \dim T_{\mathfrak{p}}$$

for any nonmaximal prime ideal \mathfrak{p} of T , and R is called *nonsingular* if the above equality holds for every prime ideal \mathfrak{p} of T . R is called a *symmetric* T -order if $\text{Hom}_T(R, T)$ is isomorphic to R as an R - R -bimodule. A finitely generated left R -module M is called (*maximal*) *Cohen-Macaulay* if it is projective as a T -module. This definition coincides with the old one when R is a commutative Cohen-Macaulay complete local ring. As before, we denote by $\text{CM}(R)$ the category of maximal Cohen-Macaulay R -modules.

Remark 6.2. Every commutative complete local Cohen-Macaulay ring containing a field is an order over a complete regular local subring ([\[Mat89\]\[29.4\]](#)). Therefore many of the rings we have been interested in so far are T -orders for some T . When R is a T -order, it sometimes happens that endomorphism rings of R -modules are also T -orders; we will apply our results in this setting as well.

6.2 n -Cluster Tilting Categories

In this section we introduce Iyama's n -cluster tilting theory; see [\[Iya08\]](#) for a more comprehensive overview.

Notation 6.3. For this section and the next, R denotes an order over a complete

regular local ring of Krull dimension d , and we always assume R is an isolated singularity.

Definition 6.4. Let \mathcal{E} be an exact category with enough projectives. For objects X, Y in \mathcal{E} we write $X \perp_n Y$ if $\text{Ext}_{\mathcal{E}}^i(X, Y) = 0$ for $0 < i \leq n$. For an exact subcategory $\mathcal{C} \subset \mathcal{E}$, put

$$\mathcal{C}^{\perp n} = \{X \in \mathcal{E} \mid M \perp_n X \text{ for all } M \in \mathcal{C}\}$$

$${}^{\perp n}\mathcal{C} = \{X \in \mathcal{E} \mid X \perp_n M \text{ for all } M \in \mathcal{C}\}$$

\mathcal{C} is called an *n-cluster tilting* subcategory of \mathcal{E} if it is functorially finite (see Definition 2.5) and $\mathcal{C} = \mathcal{C}^{\perp_{n-1}} = {}^{\perp_{n-1}}\mathcal{C}$. An object L of \mathcal{E} is called *n-cluster tilting* if $\text{add}(L)$ is an *n-cluster tilting* subcategory of \mathcal{E} .

Examples 6.5. 1. The only 1-cluster tilting subcategory of \mathcal{E} is \mathcal{E} itself, so a 1-cluster tilting object in \mathcal{E} is simply an additive generator for \mathcal{E} . In particular, $\text{CM}(R)$ has a 1-cluster tilting object if and only if R is of finite CM type.

2. The motivating example for the definition of *n-cluster tilting* comes from invariant theory. As in section 3.4, let k be a field and G a finite subgroup of $GL_d(k)$ such that G does not contain any nontrivial pseudo-reflections and $|G|$ is invertible in k . Let R be the invariant subring $k[[x_1, \dots, x_d]]^G$. If R is an isolated singularity, the R -module $k[[x_1, \dots, x_d]]$ is a $(d-1)$ -cluster tilting object in $\text{CM}(R)$ (see [Iya07b, 2.5]).

Lemma 6.6. Let $\mathcal{C} \subset \text{CM}(R)$ be a contravariantly finite subcategory with $\mathcal{C}^{\perp n} = \mathcal{C}$. If $X \in \mathcal{C}^{\perp i}$ for some $i \leq n$, then there is an exact sequence

$$0 \longrightarrow C_{n-i} \xrightarrow{f_{n-i}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $C_0, \dots, C_{n-i} \in \mathcal{C}$ such that each f_i is right minimal and

$$0 \longrightarrow (-, C_{n-i}) \xrightarrow{(-, f_{n-i})} \dots \xrightarrow{(-, f_1)} (-, C_0) \xrightarrow{(-, f_0)} (-, X) \longrightarrow 0$$

is exact on \mathcal{C} .

Proof. The proof is by decreasing induction on i . If $i = n$ then $X \in \mathcal{C}^{\perp n} \subset \mathcal{C}$ and we're done. Suppose then that $i < n$. As \mathcal{C} is functorially finite in $\mathbf{CM}(R)$, one may choose $C_0 \in \mathcal{C}$ and right minimal $f_0 : C_0 \rightarrow X$ such that $(-, C_0) \xrightarrow{(-, f_0)} (-, X) \longrightarrow 0$ is exact on \mathcal{C} . Since $R \in \mathcal{C}$, f_0 must be an epimorphism. This implies $\text{Ext}_R^1(-, \ker(f_0))$ vanishes on \mathcal{C} . Moreover from the exact sequences

$$\text{Ext}_R^j(-, X) \longrightarrow \text{Ext}_R^{j+1}(-, \ker(f_0)) \longrightarrow \text{Ext}_R^{j+1}(-, C_0)$$

and the fact that $\text{Ext}_R^j(-, X)$ and $\text{Ext}_R^{j+1}(-, C_0)$ both vanish on \mathcal{C} for $0 < j \leq i$, it follows that $\ker(f_0) \in \mathcal{C}^{\perp i+1}$. By the inductive hypothesis there is an exact sequence

$$0 \longrightarrow C_{n-i} \xrightarrow{f_n} \dots \xrightarrow{f_1} C_1 \xrightarrow{\beta} \ker(f_0) \longrightarrow 0$$

with $C_1, \dots, C_{n-i} \in \mathcal{C}$ such that

$$0 \longrightarrow (-, C_{n-i}) \xrightarrow{(-, f_n)} \dots \xrightarrow{(-, f_1)} (-, C_1) \xrightarrow{(-, \beta)} (-, \ker(f_0)) \longrightarrow 0$$

is exact on \mathcal{C} . Setting $f_0 = \alpha\beta$ yields the desired resolution of X . \square

Corollary 6.7. *Let \mathcal{C} be an n -cluster tilting subcategory of $\mathbf{CM}(R)$. For any object X of $\mathbf{CM}(R)$, there is an exact sequence*

$$0 \longrightarrow C_{n-1} \xrightarrow{f_n} \dots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$$

such that

$$0 \longrightarrow (-, C_{n-1}) \xrightarrow{(-, f_n)} \dots \xrightarrow{(-, f_1)} (-, C_0) \xrightarrow{(-, f_0)} (-, X) \longrightarrow 0$$

is a minimal projective resolution of $(-, X)$ in $\mathbf{mod}(\mathcal{C})$.

Remark 6.8. In particular, Corollary 6.7 states when $\mathcal{C} \subset \text{CM}(R)$ is n -cluster tilting, every maximal Cohen-Macaulay module has a bounded resolution in \mathcal{C} . This suggests that the inclusion functor $\mathcal{C} \rightarrow \text{CM}(R)$ might induce an essentially surjective functor on bounded derived categories. I have not been able to prove this. In particular, Keller's criterion 4.6 does not apply here.

However, Corollary 6.7 does imply the Yoneda functor $\mathcal{C}^\oplus \rightarrow \text{mod}(\mathcal{C})$ is a derived equivalence, as we now prove.

Proposition 6.9. *Let $\mathcal{C} \subset \text{CM}(R)$ be an n -cluster tilting subcategory. Then the Yoneda functor $\mathcal{C}^\oplus \rightarrow \text{mod}(\mathcal{C})$, $M \mapsto \mathcal{C}(-, M)$, is a derived equivalence. In particular, since $\text{CM}(R)$ is trivially a 1-cluster tilting subcategory of itself, $\text{CM}(R)^\oplus \rightarrow \text{mod}(\text{CM}(R))$ is a derived equivalence.*

Proof. We use Keller's criterion 4.6. Let $F' \twoheadrightarrow F \xrightarrow{p} h(M)$ be a short exact sequence of finitely presented functors on \mathcal{C} . Then since $h(M)$ is projective, p has a right inverse i . Since the following diagram commutes, Keller's criterion is satisfied.

$$\begin{array}{ccccc} 0 & \twoheadrightarrow & h(M) & \xrightarrow{\text{id}} & h(M) \\ \downarrow & & \downarrow i & & \parallel \\ F' & \twoheadrightarrow & F & \xrightarrow{p} & h(M) \end{array}$$

Therefore $D^b(h) : D^b(\mathcal{C}^\oplus) \rightarrow D^b(\text{mod}(\mathcal{C}))$ is fully faithful. The essential image of $D^b(h)$ is a full subcategory of $D^b(\text{mod}(\mathcal{C}))$ closed under cones and suspensions, and it contains all complexes concentrated in degree zero by Corollary 6.7. Therefore $D^b(h)$ is essentially surjective. \square

Remark 6.10. One can deduce Corollary 4.18 from a theorem of Schlichting when $\mathcal{A} = \mathcal{C} \subset \text{CM}(R)$ is an n -cluster tilting subcategory. According to [Sch06, Proposition 2], $K(\text{mod}(\mathcal{C})_0)$ is the homotopy fiber of $K(\text{id}) : K(\mathcal{C}^\oplus) \rightarrow K(\text{mod}(\mathcal{C}))$. Now Corollary 4.18 follows from the fact that $h : \mathcal{C}^\oplus \rightarrow \text{mod}(\mathcal{C})$ and

the inclusion $\mathcal{C} \rightarrow \text{mod}(R)$ are K -theory equivalences (the former by Proposition 6.9, the latter by the resolution theorem 4.5).

6.3 Higher Auslander-Reiten Theory

In this section we explain the main theorems of higher Auslander-Reiten theory ([Iya07b]). This theory generalizes the classical Auslander-Reiten theory of 3.2.

Adopt the notation from 6.3, and let $\mathcal{C} \subset \text{CM}(R)$ be an n -cluster tilting subcategory.

Proposition 6.11. *For any indecomposable nonprojective object X of $\text{CM}(R)$, there is an exact sequence*

$$0 \longrightarrow C_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$$

such that

$$0 \longrightarrow (-, C_n) \xrightarrow{(-, f_n)} \cdots \xrightarrow{(-, f_1)} (-, C_0) \xrightarrow{(-, f_0)} (-, X)$$

is a minimal projective resolution of $(-, X)/\text{rad}_{\text{CM}(R)}(-, X)$ in $\text{mod}(\mathcal{C})$.

Proof. There is a minimal right almost split map $f : Z \rightarrow X$ in $\text{CM}(R)$; this is 3.20 if R is commutative CM local, or [Aus86a][Main Theorem] for the general case. Choose a right \mathcal{C} -approximation $g : C \rightarrow Z$ of Z , and let $f_0 : C_0 \rightarrow X$ be a right minimal version of gf . Then $(-, f_0)$ is a projective cover of $\text{rad}_{\mathcal{C}}(-, X)$. By Corollary 6.7, there is a complex

$$0 \longrightarrow C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1'} \ker(f_0) \longrightarrow 0$$

with all maps right minimal and such that

$$0 \longrightarrow (-, C_n) \xrightarrow{(-, f_n)} \cdots \xrightarrow{(-, f_2)} (-, C_1) \xrightarrow{(-, f_1')} (-, \ker(f_0)) \longrightarrow 0$$

is exact. Setting f_1 to be the composition of f'_1 with the inclusion $\ker(f_0) \rightarrow C_0$ yields the desired complex. \square

A sequence as in Proposition 6.11 is called an n -Auslander-Reiten sequence. Note that a 1-Auslander-Reiten sequence is just an Auslander-Reiten sequence.

The next theorem follows easily from results of Auslander; our proof mimics one in [Yos90].

Theorem 6.12. *Suppose \mathcal{C} has an additive generator. Then every functor F in $\mathbf{mod}_0(\mathcal{C})$ admits a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

with F_i in $\mathbf{mod}_0(\mathcal{C})$ and F_i/F_{i-1} simple in $\mathbf{Mod}(\mathcal{C})$ for all i .

Proof. F has a presentation of the form

$$\mathcal{C}(-, N) \xrightarrow{\mathcal{C}(-, f)} \mathcal{C}(-, M) \longrightarrow F \longrightarrow 0$$

for some epimorphism $f : N \twoheadrightarrow M$ in \mathcal{C} . Set $K = \ker(f)$, so F is a subfunctor of $\mathrm{Ext}_R^1(-, K)$. For any module L in \mathcal{C} and prime $\mathfrak{p} \neq \mathfrak{m}$, $L_{\mathfrak{p}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$ -module; since $R_{\mathfrak{p}}$ is regular local, $L_{\mathfrak{p}}$ is in fact a free $R_{\mathfrak{p}}$ -module. Therefore $(\mathrm{Ext}_R^1(L, K))_{\mathfrak{p}} = \mathrm{Ext}_{R_{\mathfrak{p}}}^1(L_{\mathfrak{p}}, K_{\mathfrak{p}}) = 0$. Since $\mathrm{Ext}_R^1(L, K)$ is supported only at \mathfrak{m} , it must be a finite length R -module. Therefore the submodule $F(L)$ is finite length as well.

Now let L be an additive generator for \mathcal{C} . The proof proceeds by induction on the length of $F(L)$. If $\mathrm{length}(F(L)) = 0$, F vanishes on $\mathrm{add}(L) = \mathcal{C}$, so $F \cong 0$ and F trivially admits the desired filtration. Suppose, then, that $\mathrm{length}(F(L)) > 0$, and assume that any functor G in $\mathbf{mod}_0(\mathcal{C})$ with $\mathrm{length}(G(L)) < \mathrm{length}(F(L))$ admits a filtration as above. Choose an indecomposable M in \mathcal{C} with $F(M) \neq 0$;

then choose an epimorphism of R_M -modules $p : F(M) \longrightarrow \kappa_M^{\text{op}}$. p extends to a natural transformation $\pi : F \longrightarrow S_M$. Let $G = \ker(\pi)$, so there is an exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow S_M \longrightarrow 0.$$

Therefore $\text{length}(G(L)) < \text{length}(F(L))$. Since G is a subfunctor of F , G is in $\text{mod}_0(\mathcal{C})$, so G admits the desired filtration. From the exact sequence above, it follows that F admits such a filtration as well. \square

The next proposition is a triangulated version of Theorem 6.12.

Proposition 6.13. *Let $\mathcal{C} \subset \text{CM}(R)$ be an n -cluster tilting subcategory, and assume \mathcal{C} has an additive generator L . Let $\text{Ho}(\text{Ac}^b(\mathcal{C}))$ be homotopy category of the category of bounded acyclic complexes in \mathcal{C} . (Note that by Lemma 3.9, a bounded complex in \mathcal{C} is acyclic in the sense of Definition 4.3 if and only if it is acyclic as a complex of R -modules.)*

Then $\text{Ho}(\text{Ac}^b(\mathcal{C}))$ is generated, as a triangulated category, by the n -Auslander-Reiten sequences.

Proof. Let

$$Y : \quad \cdots \longrightarrow Y_{i-1} \xrightarrow{d_{i-1}} Y_i \xrightarrow{d_i} Y_{i+1} \longrightarrow \cdots$$

be a bounded acyclic complex in \mathcal{C} . Denote by $(-, Y)$ the corresponding complex in $\text{mod}(\mathcal{C})$

$$(-, Y) : \quad \cdots \longrightarrow (-, Y_{i-1}) \xrightarrow{(-, d_{i-1})} (-, Y_i) \xrightarrow{(-, d_i)} (-, Y_{i+1}) \longrightarrow \cdots$$

Let H_Y^i be the homology of $(-, Y)$ at $(-, Y_i)$, i.e. $H_Y^i = \ker(-, d_i) / \text{im}(-, d_{i-1})$. Then $H_Y^i(R) = 0$ so $H_Y^i \in \text{mod}_0(\mathcal{C})$. By Theorem 6.12, H_Y^i is of finite length in $\text{Mod}(\mathcal{C})$. Define $l_Y = \sum \text{length}(H_Y^i)$. We proceed by induction on l_Y .

First we prove that when $l_Y = 0$, Y is contractible, so Y is certainly in $\text{Ho}(\text{Ac}^b(\mathcal{C}))$. This requires a separate induction on the number of nonzero terms in the complex Y . Let i be the maximum integer such that $Y_{i+1} \neq 0$.

$$(-, Y_i) \xrightarrow{(-, d_i)} (-, Y_{i+1}) \longrightarrow 0$$

is exact on \mathcal{C} and $Y_{i+1} \in \mathcal{C}$, so there is $s : Y_{i+1} \rightarrow Y_i$ such that $d_i s = \text{id}_{Y_{i+1}}$. Y is therefore a direct sum of the complex

$$\cdots \longrightarrow Y_{i-2} \xrightarrow{d_{i-1}} Y_{i-1} \xrightarrow{d_i} \text{coker}(s) \longrightarrow 0$$

and a shift of the complex

$$0 \longrightarrow Y_i \xlongequal{\quad} Y_i \longrightarrow 0.$$

Since s is a split monomorphism, $\text{coker}(s) \in \text{mod}(\mathcal{C})$ so the first complex is contractible by induction. Therefore Y is contractible.

Suppose now that $l_Y > 0$, and choose i such that $H_Y^i \neq 0$. By the classification of simple functors (2.16), H_Y^i must have a subfunctor S_M for some indecomposable nonprojective $M \in \mathcal{C}$. Let

$$C_M : \quad 0 \longrightarrow C_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \longrightarrow 0$$

be the n -Auslander-Reiten sequence ending in M , considered as a complex with M in degree i , and let $(-, C_M)$ be the associated complex

$$0 \longrightarrow (-, C_n) \xrightarrow{(-, f_n)} \cdots \xrightarrow{(-, f_1)} (-, C_0) \xrightarrow{(-, f_0)} (-, M)$$

in $\text{mod}(\mathcal{C})$. Since $(-, C_M)$ is a projective resolution of (a shift of) S_M in $\text{mod}(\mathcal{C})$, the inclusion $S_M \hookrightarrow H_i \hookrightarrow (-, Y_{i+1})$ lifts to a map of complexes $(-, g) : (-, C_M) \rightarrow (-, Y)$, induced by some map $g : C_M \rightarrow Y$. Let $Z = \text{cone}(g)$, a complex in \mathcal{C} . By the long exact sequence in homology associated to the mapping cone sequence $(-, C_M) \xrightarrow{(-, g)} (-, Y) \longrightarrow (-, Z)$, one sees that $l_Z = l_Y - 1$. By induction, Z is in $\text{Ho}(\text{Ac}^b(\mathcal{C}))$, so Y is as well. \square

6.4 Noncommutative Crepant Resolutions

As before, let d be the Krull dimension of the base commutative ring. $(d - 1)$ -cluster tilting subcategories have nice homological properties, as we explain below.

Definition 6.14. A module-finite R -algebra Λ is called n -Calabi-Yau or n -CY if there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(X, Y[n]) \cong D(\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(Y, X))$$

for $X, Y \in \mathcal{D}^b(\mathrm{fl}\Lambda)$. Λ is called n -CY⁻ if there is such an isomorphism for any $X \in \mathcal{D}^b(\mathrm{fl}(\Lambda))$ and $Y \in \mathrm{Ho}^b(\mathrm{proj}(\Lambda))$.

Theorem 6.15 ([Iya07a, 4.7], [IR08, Theorem 3.2]). *If M is a $(d - 1)$ -cluster tilting object of $\mathrm{CM}(R)$ then $\mathrm{End}_R(M)$ is a nonsingular R -order. If in addition $\mathrm{End}_R(M)$ is a symmetric R -algebra, it is d -CY.*

The following theorem follows from Proposition 2.4(3) and Theorem 3.2(2) in [IR08].

Theorem 6.16. *Let R be a d -dimensional Gorenstein ring and $M \in \mathrm{CM}(R)$. Then $\mathrm{End}_R(M)$ is d -CY⁻.*

$(d - 1)$ -cluster tilting objects are closely related to Van den Bergh's noncommutative crepant resolutions, which we now define.

Definition 6.17. We say M gives a *noncommutative crepant resolution (NCCR)* $\mathrm{End}_R(M)$ of R if

1. the natural map $M \rightarrow \mathrm{Hom}_T(\mathrm{Hom}_T(M, T), T)$ is an isomorphism;
2. $\mathrm{End}_R(M)$ is a nonsingular R -order; and

3. $M_{\mathfrak{p}}$ is a generator of $R_{\mathfrak{p}}$ (i.e., $R_{\mathfrak{p}} \in \mathbf{add}_{\mathbf{mod}(R_{\mathfrak{p}})}(M_{\mathfrak{p}})$) for each height one prime ideal \mathfrak{p} of T .

Condition 3 is automatically satisfied if R is a commutative normal domain, but it seems to be necessary when R is not commutative (see [IR08, §8]).

Noncommutative crepant resolutions are closely related to $(d-1)$ -cluster tilting objects. The next theorem states the relationship precisely.

Theorem 6.18 ([Iya07b]). *The following are equivalent.*

1. M is a $(d-1)$ -cluster tilting object in $\mathbf{CM}(R)$.
2. M is a generator-cogenerator in $\mathbf{CM}(R)$ and M gives an NCCR of R .

Noncommutative crepant resolutions have been studied extensively in recent years, partly because of the role they play in the resolution of singularities in a noncommutative version of the minimal model program from algebraic geometry. See [Leu12, § K] for a good explanation of this role. Here we only note that the bounded derived categories of NCCRs are of great interest (see e.g. [IR08, Corollary 8.8]), and that we will prove results about the K -theory of a NCCR (see (7.4)), which should be closely related to its derived category.

7 K -Theory of CM Modules

7.1 The Long Exact Sequence

Fix a Henselian Cohen-Macaulay local ring R with maximal ideal \mathfrak{m} . Assume also that R has a canonical module, and that R is an isolated singularity. As R is Henselian local, $\mathbf{mod}(R)$ is a Krull-Schmidt category by the Krull-Schmidt theorem (2.12).

Let $\mathcal{C} \subset \mathbf{CM}(R)$ be an n -cluster tilting subcategory. Let $r : \mathbf{mod}(\mathcal{C}) \rightarrow \mathbf{mod}(R)$ be the evaluation functor $F \mapsto F(R)$. Let $\mathbf{mod}_0(\mathcal{C}) = \ker(r) \subset \mathbf{mod}(\mathcal{C})$, i.e. $\mathbf{mod}_0(\mathcal{C})$ is the category of finitely presented functors $F : \mathcal{C}^{\text{op}} \rightarrow (\text{abelian groups})$ satisfying $F(R) \cong 0$. $\mathbf{proj}(R) \subset \mathcal{C}$ is a contravariantly finite subcategory, and r is the composition of the restriction functor $\mathbf{mod}(\mathcal{C}) \rightarrow \mathbf{mod}(\mathbf{proj}(R))$ with the equivalence $\mathbf{mod}(\mathbf{proj}(R)) \rightarrow \mathbf{mod}(R)$, $F \mapsto F(R)$. Therefore by Corollary 4.18, the following sequence of exact categories induces a homotopy fiber sequence of K -theory spectra.

$$\mathbf{mod}_0(\mathcal{C}) \longrightarrow \mathbf{mod}(\mathcal{C}) \xrightarrow{r} \mathbf{mod}(R)$$

In this section we study the first two terms of the sequence.

Definition 7.1. Let $\mathbf{mod}_0^s(\mathcal{C})$ be the full subcategory of $\mathbf{mod}_0(\mathcal{C})$ consisting of objects which are semisimple in $\mathbf{Mod}(\mathcal{C})$.

By Proposition 2.16 and Remark 3.21, $\mathbf{mod}_0^s(\mathcal{C})$ consists of those functors which are finite direct sums of the functors S_M .

Proposition 7.2. *If \mathcal{C} has an additive generator, the inclusion $\mathbf{mod}_0^s(\mathcal{C}) \rightarrow \mathbf{mod}_0(\mathcal{C})$ is a K -theory equivalence.*

Proof. Since subobjects, quotients, and products of semisimple objects in an abelian category are again semisimple, $\mathbf{mod}_0^s(\mathcal{C})$ is closed under taking subobjects, quotients, and products. Theorem 6.12 allows us to apply Dévissage to the subcategory $\mathbf{mod}_0^s(\mathcal{C}) \subset \mathbf{mod}_0(\mathcal{C})$. The conclusion follows. \square

Assume \mathcal{C} has an additive generator L . Let $\mathbf{ind}(\mathcal{C})$ denote the set of isomorphism classes of indecomposable objects in \mathcal{C} , and put $\mathbf{ind}_0(\mathcal{C}) = \mathbf{ind}(\mathcal{C}) \setminus \{[R]\}$. $\mathbf{mod}_0^s(\mathcal{C})$ is semisimple, and by Proposition 2.16, its simple objects are the

functors S_M for $[M] \in \text{ind}_0(\mathcal{C})$. Since $\text{End}_{\text{mod}_0^s(\mathcal{C})}(S_M)$ is the quotient of $\text{End}_{\text{mod}(\mathcal{C})}(\mathcal{C}(-, M)) = \text{End}_R(M)$ by its Jacobson radical, $\text{End}_{\text{mod}_0^s(\mathcal{C})}(S_M) = \kappa_M^{\text{op}}$.

Therefore the equivalences

$$\text{proj}(\kappa_M) = \text{proj}(\text{End}(S_M)^{\text{op}}) \simeq \text{add}(S_M)$$

induce an equivalence

$$\bigoplus_{[M] \in \text{ind}_0(\mathcal{C})} \text{proj}(\kappa_M) \xrightarrow{\sim} \text{mod}_0^s(\mathcal{C}).$$

Put $\Lambda = (\text{End}_R L)^{\text{op}}$. Λ is sometimes called the *Auslander algebra*. Since L is an additive generator for $\text{CM}(R)^\oplus$, the horizontal functors in the diagram below are equivalences:

$$\begin{array}{ccc} \mathcal{C}^\oplus & \xrightarrow[\simeq]{\mathcal{C}^\oplus(L, -)} & \text{proj}(\Lambda) \\ \downarrow h & & \downarrow \\ \text{mod}(\mathcal{C}) & \xrightarrow[\simeq]{F \mapsto F(L)} & \text{mod}(\Lambda) \end{array}$$

Combining everything, we end up with a diagram

$$\begin{array}{ccccc} \bigoplus_{\text{ind}_0(\mathcal{C})} \text{proj}(\kappa_M) & & & & \text{proj}(\Lambda) & (7.3) \\ \simeq \downarrow & & & \swarrow \simeq & \downarrow \simeq_K & \\ \text{mod}_0^s(\mathcal{C}) & & & \mathcal{C}^\oplus & \text{mod}(\Lambda) & \\ \simeq_K \downarrow & & \swarrow \simeq & \downarrow \simeq_K & & \\ \text{mod}_0(\mathcal{C}) & \longrightarrow & \text{mod}(\mathcal{C}) & \longrightarrow & \text{mod}(R) & \end{array}$$

in which arrows labeled \simeq_K induce equivalences in K -theory and the bottom horizontal row induces a homotopy fiber sequence in K -theory. Therefore there are homotopy fiber sequences

$$\bigvee_{[M] \in \text{ind}_0(\mathcal{C})} K(\kappa_M) \longrightarrow K'(\Lambda) \longrightarrow K'(R) \quad (7.4)$$

$$\bigvee_{[M] \in \text{ind}_0(\mathcal{C})} K(\kappa_M) \xrightarrow{\alpha} K(\mathcal{C}^\oplus) \xrightarrow{\beta} K'(R) \quad (7.5)$$

Taking homotopy groups in (7.4) yields the long exact sequence of Theorem 1.3.

In particular, 7.5 shows that $K'_0(R)$ is a finitely generated abelian group when R is a Henselian CM local ring which has a maximal Cohen-Macaulay module giving an NCCR. This gives a special case of the following theorem of Dao-Iyama-Takahashi-Vial, which was proved using seemingly different techniques.

Theorem 7.6 ([DIT12, Theorem 2.3]). *Let S be a semilocal ring, and suppose S has an NCCR. Then $K'_0(S)$ is a finitely generated abelian group.*

Remark 7.7. By Nakayama's Lemma, the image of \mathfrak{m} in R_M is contained in the maximal ideal of R_M . So we may view κ_M as a division algebra over R/\mathfrak{m} . As R_M is a finitely generated R -module, κ_M is a finite dimensional vector space over R/\mathfrak{m} . In particular, if R/\mathfrak{m} is algebraically closed, $\kappa_M = R/\mathfrak{m}$.

Remark 7.8. Suppose R is of the form $S/(w)$ for some regular local ring S and $w \in S$. Then we may apply the techniques above to obtain a decomposition of the K -theory of the category \mathbf{MF} of matrix factorizations in S with potential w . An object of this category is a $\mathbb{Z}/2\mathbb{Z}$ -graded finitely generated free S -module X with a degree-one endomorphism d_X such that $d_X^2 = w \cdot \text{id}$. A morphism $f : X \rightarrow Y$ in \mathbf{MF} is a degree zero map satisfying $d_Y f = f d_X$.

\mathbf{MF} is a Frobenius category whose subcategory $\text{prinj}(\mathbf{MF})$ of projective-injective objects consists of the contractible matrix factorizations – that is, those objects X for which there is a degree-one endomorphism $t : X \rightarrow X$ such that $t d_X + d_X t = \text{id}_X$. Any Frobenius category \mathcal{F} defines an exact category with weak equivalences $w\mathcal{F}$ consisting of those morphisms becoming invertible in the stable category $\mathcal{F}/\text{prinj}(\mathcal{F})$. We shall take the K -theory of \mathbf{MF} relative to this subcategory $w\mathbf{MF}$ of weak equivalences.

The category $\text{Ch}^b(\mathcal{E})$ of bounded chain complexes in an exact category \mathcal{E} is a Frobenius category; its conflations are the sequences which are degree-wise split, and its projective-injective objects are the contractible complexes. Denote by $\text{perf}(R)$ the category of perfect complexes of R -modules, i.e. the exact subcategory of $\text{Ch}^b\text{mod}(R)$ consisting of complexes quasi-isomorphic to a complex of free R -modules. There is a map of Frobenius pairs

$$\Omega : (\text{MF}, \text{prinj}(\text{MF})) \longrightarrow (\text{Ch}^b\text{mod}(R), \text{perf}(R))$$

taking a matrix factorization $X^1 \begin{smallmatrix} \xrightarrow{d^1} \\ \xleftarrow{d^0} \end{smallmatrix} X^0$ to $\text{coker}(d^1)$, considered as a complex concentrated in degree zero. (Since $\text{coker}(d^1)$ is annihilated by w , we may view it as an R -module.) By [Orl04, Theorem 3.9], Ω induces an equivalence on derived categories, so by Waldhausen approximation 4.13.2 it is an equivalence in K -theory. Since the degree-zero inclusions $\text{mod}(R) \rightarrow \text{Ch}^b\text{mod}(R)$ and $\text{proj}(R) \rightarrow \text{Ch}^b\text{proj}(R)$ are K -theory equivalences ([TT90, 1.11.7]), and since the inclusion $\text{Ch}^b\text{proj}(R) \rightarrow \text{perf}(R)$ is a derived equivalence, it follows that $K(\text{MF})$ is equivalent to the homotopy cofiber of the map $K(R) \rightarrow K'(R)$ induced by the inclusion $\text{proj}(R) \rightarrow \text{mod}(R)$.

Let $r : \text{proj}(R) \rightarrow \text{CM}(R)^\oplus$ be the inclusion, and set $X = \text{cone}(K(r))$. Consider the following exact triangles of spectra.

$$\begin{array}{ccccccc} K(R) & \xrightarrow{K(r)} & K(\text{CM}(R)^\oplus) & \xrightarrow{\rho} & X & \longrightarrow & \sum K(R) \\ K(\text{CM}(R)^\oplus) & \xrightarrow{\beta} & K'(R) & \longrightarrow & \sum_{\text{ind}_0(\text{CM}(R))} \bigvee K(\kappa_M) & \xrightarrow{\sum \alpha} & \sum K(\text{CM}(R)^\oplus) \\ K(R) & \xrightarrow{\beta \circ K(r)} & K'(R) & \longrightarrow & K(\text{MF}) & \longrightarrow & \sum K(R) \end{array} \quad (7.9)$$

The middle sequence is the triangle from (7.5), rotated once. Using the octahedral

axiom to compare the cones of β , $K(r)$, and $\beta \circ K(r)$, we obtain an exact triangle

$$\bigvee_{\text{ind}_0(\text{CM}(R))} K(\kappa_M) \xrightarrow{\alpha'} X \longrightarrow K(\text{MF}) \longrightarrow \sum_{\text{ind}_0(\text{CM}(R))} \bigvee_{\text{ind}_0(\text{CM}(R))} K(\kappa_M). \quad (7.10)$$

We will study α' in the next section.

7.2 The Auslander-Reiten Matrix

In this section, assume that $k = R/\mathfrak{m}$ is algebraically closed and that R contains k . Let $\mathcal{C} \subset \text{CM}(R)$ be an n -cluster tilting object with additive generator L . We wish to understand the map

$$\alpha : \bigvee_{\text{ind}_0(\mathcal{C})} K(k) \longrightarrow K(\mathcal{C}^\oplus) \quad (7.11)$$

which appears in (7.5).

Let M^0, \dots, M^t be the indecomposable objects of \mathcal{C} , with $M^0 = R$. For $j > 0$, set

$$0 \longrightarrow C_n^j \longrightarrow \dots \longrightarrow C_0^j \longrightarrow M^j \longrightarrow 0$$

to be the n -Auslander-Reiten sequence ending in M^j . Given any Q in \mathcal{C} , let $\#(j, Q)$ be the number of M^j -summands appearing in a decomposition of Q into indecomposables.

Denote by k_j the object of $\bigoplus_{\text{ind}_0(\mathcal{C})} \text{mod}(k)$ which is k in the M^j coordinate and 0 in the others. Note that to define a k -linear functor out of $\bigoplus_{\text{ind}_0(\mathcal{C})} \text{mod}(k)$, one needs only to specify the image of each object k_j .

Set $a : \bigoplus_{\text{ind}_0(\mathcal{C})} \text{mod}(k) \rightarrow \text{mod}(\mathcal{C})$ to be the k -linear functor sending k_j to S_{M^j} , and as before let $h : \mathcal{C}^\oplus \rightarrow \text{mod}(\mathcal{C})$ be the Yoneda functor. Tracing through the functors in (7.3), one sees that $K(a) = K(h) \circ \alpha$. Define k -linear functors

$$a_i : \bigoplus_{\text{ind}_0(\mathcal{C})} \text{mod}(k) \rightarrow \mathcal{C}^\oplus \quad (0 \leq i \leq n+1)$$

$$\text{by } \begin{cases} a_i(k_j) = C_{i-1}^j & (1 \leq i \leq n+1) \\ a_0(k_j) = M^j \end{cases}$$

The functors a_i are defined so that there is an exact sequence of functors

$$0 \longrightarrow h \circ a_{n+1} \longrightarrow \cdots \longrightarrow h \circ a_0 \longrightarrow a \longrightarrow 0.$$

Therefore by the additivity theorem (4.4),

$$\sum_{i=0}^{n+1} (-1)^i K(h \circ a_i) = K(a)$$

so that

$$\alpha = K(h)^{-1} K(a) = \sum_{i=0}^{n+1} (-1)^i K(a_i). \quad (7.12)$$

Let $m_l : \text{mod}(k) \rightarrow \mathcal{C}^\oplus$ be the k -linear functor which sends k to M^l . Form a $(t+1) \times t$ integer matrix T whose lj -entry is $\sum_{i=0}^n (-1)^i \#(l, C_i^j)$. (T has a 0th row but no 0th column.) We call T the *Auslander-Reiten matrix*. Applying the additivity theorem to each a_i , we conclude from (7.12) that the j th component of α is

$$(\alpha)_j = \sum_l T_{lj} K(m_l). \quad (7.13)$$

This shows that α can be described concisely as in the following proposition.

Proposition 7.14. *Let $m = \bigoplus_{\text{ind}(\mathcal{C})} m_l : \bigoplus \text{mod}(k) \rightarrow \mathcal{C}^\oplus$, and let T be the Auslander-Reiten matrix. Then*

$$\alpha = K(m) \circ (T \cdot \text{id}_{K(k)}).$$

Identifying $K_0(\mathcal{C}^\oplus)$ with \mathbb{Z}^{t+1} via the basis $\{[M_0], \dots, [M_t]\}$, we see that $K_0(m_l) : K_0(\text{mod}(k)) = \mathbb{Z} \rightarrow \mathbb{Z}^{t+1}$ is just the inclusion into the l th coordinate. Therefore $\pi_0(\alpha)$, as a map between free abelian groups, is defined by T . This is the description of $\pi_0(\alpha)$ originally given in [AR86, § 4.3].

T has the following alternative description:

Proposition 7.15. *Let $\Lambda = \text{End}(L)^{\text{op}}$, let $P_0^j = (L, M^j) \in \text{proj}(\Lambda)$, and let $S^j = P_0^j/\text{rad}(P_0^j)$, so P_0^0, \dots, P_0^t are the indecomposable projective left Λ -modules and S_0, \dots, S_t are the simple left Λ -modules. Then*

$$T_{lj} = \sum_{i=0}^{n+1} (-1)^i \dim(\text{Ext}_{\Lambda}^i(S^j, S^l)).$$

To prove this, we first need the following lemma.

Lemma 7.16. *Suppose Λ is a semiperfect ring and there are maps in $\text{mod}(\Lambda)$*

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_0 \\ p_1 \downarrow & \swarrow p_0 & \\ S & & \end{array}$$

with P_1 and P_0 projective and S simple. If P_0 is a projective cover of $\text{coker}(f)$, then $p_1 = 0$.

Proof. Suppose $p_1 \neq 0$. Let $n : N \rightarrow S$ be a projective cover of S . Then for $i = 0, 1$ there is a commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{s_i} & P_i & \xrightarrow{r_i} & N \\ & \searrow n & \downarrow p_i & \swarrow n & \\ & & S & & \end{array}$$

with s_i a split monomorphism and r_i a split epimorphism. Then $n = p_1 s_1 = p_0 f s_1 = n r_0 f s_1$ so $r_0 f s_1$ is an isomorphism. It follows that $\text{im}(f)$ contains the summand $\text{im}(f s_1)$ of P_0 . Let $p : P_0 \rightarrow \text{coker}(f)$ be the canonical map; then $p s_1 f r_0 = p$, contradicting that p is right minimal. \square

Corollary 7.17. *Let Λ be a semiperfect ring and $(P_{\bullet}, d_{\bullet})$ a minimal projective resolution of a module $M \in \text{mod}(\Lambda)$. Let $S \in \text{mod}(\Lambda)$ be a simple module, and let $\text{Hom}_{\Lambda}(P_{\bullet}, S)$ be the complex obtained by applying $\text{Hom}_{\Lambda}(-, S)$ to P_{\bullet} . Then $B^i \text{Hom}_{\Lambda}(P_{\bullet}, S) = 0$ and $Z^i \text{Hom}_{\Lambda}(P_{\bullet}, S) = \text{Hom}_{\Lambda}(P_i, S)$. Consequently*

$$\text{Ext}_{\Lambda}^i(M, S) = \text{Hom}_{\Lambda}(P_i, S).$$

Proof. This follows directly from Lemma 7.16. \square

Proof of Proposition 7.15. We need to show $\#(l, C_i^j) = \text{length}(\text{Ext}_\Lambda^i(S^j, S^l))$. For $i > 0$, put $P_i^j := (L, C_{i+1}^j)$.

$$0 \longrightarrow (-, C_n^j) \longrightarrow \cdots \longrightarrow (-, C_0^j) \longrightarrow (-, M^j)$$

is a minimal projective resolution of the simple functor $(-, M^j)/\text{rad}(-, M^j)$ in $\text{mod}(\mathcal{C})$. The functor $F \mapsto F(L) : \text{mod}(\mathcal{C}) \rightarrow \text{mod}(\Lambda)$ is an equivalence, so

$$0 \longrightarrow P_{n+1}^j \longrightarrow \cdots \longrightarrow P_1^j \longrightarrow P_0^j$$

is a minimal projective resolution of S^j in $\text{mod}(\Lambda)$. Therefore by Corollary 7.17, $\text{Ext}_\Lambda^i(S^j, S^l) = \text{Hom}_\Lambda(P_i^j, S^l)$. Decomposing P_i^j into indecomposables, one sees easily that $\text{length}(\text{Hom}_\Lambda(P_i^j, S^l))$ is the multiplicity of the projective cover P_0^l of S^l as a summand of P_i^j . This multiplicity equals the multiplicity of M_l as a summand of C_i^j , which is exactly $\#(l, C_i^j)$. \square

Let T' be the $t \times t$ integer matrix obtained from T by deleting its top row, which corresponds to the indecomposable $M_0 = R$. We call T' the *stable Auslander-Reiten matrix*. Just as T described α , T' describes the map $\alpha' = \rho \circ \alpha : \bigvee_{\text{ind}_0(\mathcal{C})} K(k) \rightarrow X$ from (7.10). Recall from (7.9) the homotopy fiber sequence

$$K(R) \xrightarrow{K(r)} K(\mathcal{C}^\oplus) \xrightarrow{\rho} X$$

Since $m_0 : \text{mod}(k) \rightarrow \mathcal{C}^\oplus$ factors through r , $\rho \circ K(m_0)$ is nullhomotopic and therefore the j th component of α' is given by

$$\begin{aligned} (\alpha')_j &= \rho \circ \alpha_j \\ &= \rho \circ \sum_{l \geq 0} T_{lj} K(m_l) \\ &= \rho \circ \sum_{l > 0} T'_{lj} K(m_l) \end{aligned} \tag{7.18}$$

This proves the following.

Proposition 7.19. *Let $m' = \bigoplus_{l>0} m_l : \bigoplus_{\text{ind}_0(\mathcal{C})} \text{mod}(k) \rightarrow \mathcal{C}^\oplus$, and let T' be the stable Auslander-Reiten matrix. Then*

$$\alpha' = \rho \circ K(m') \circ (T' \cdot \text{id}_{K(k)}).$$

7.3 An Example

Let R be a 1-dimensional singularity of type A_{2n} , i.e. $R = k[[t^2, t^{2n+1}]]$ with k an algebraically closed field. The MCM R -modules are the modules $M_i = k[[t^2, t^{2(n-i)+1}]]$, $i = 0, \dots, n$, on which R acts by multiplication. The Auslander-Reiten quiver of R is then

$$[R] \begin{array}{c} \xleftarrow[t^2]{} \\ \xrightarrow[t^2]{} \end{array} [M_1] \begin{array}{c} \xleftarrow[t^2]{} \\ \xrightarrow[t^2]{} \end{array} \cdots \begin{array}{c} \xleftarrow[t^2]{} \\ \xrightarrow[t^2]{} \end{array} [M_n] \begin{array}{c} \xleftarrow[t^2]{} \\ \xrightarrow[t^2]{} \end{array} \Big) t \quad (7.20)$$

(each right arrow is the inclusion map). In particular, R is of finite Cohen-Macaulay type, so any additive generator for $\text{CM}(R)$ is a 1-cluster tilting object. In this section we will try to describe, as explicitly as possible, the groups $K'_1(R)$ and $K_1(\text{MF})$, using the techniques developed elsewhere in this paper. These descriptions appear in Proposition 7.26.

Let $\mathcal{B}_i = \text{add}(M_0, \dots, M_i) \subset \mathcal{C}^\oplus$, and let $\mathcal{B}_{-1} = \{0\} \subset \mathcal{C}^\oplus$. Let $f_i : \mathcal{B}_{i-1} \rightarrow \mathcal{B}_i$ denote the inclusion functor and $p_i : \mathcal{B}_i \rightarrow \mathcal{B}_i/\mathcal{B}_{i-1}$ the quotient functor; let F_i denote the image in $K_1(\mathcal{C}^\oplus)$ of $K_1(\mathcal{B}_i)$. Then the solid diagram below commutes and has exact rows; the top row is exact by Theorem 5.12.

$$\begin{array}{ccccccc} K_1(\mathcal{B}_{i-1}) & \xrightarrow{K_1(f_i)} & K_1(\mathcal{B}_i) & \xrightarrow{K_1(p_i)} & K_1(\mathcal{B}_i/\mathcal{B}_{i-1}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & \nearrow & \downarrow & & \\ 0 & \longrightarrow & F_{i-1} & \longrightarrow & F_i & \longrightarrow & F_i/F_{i-1} \longrightarrow 0 \end{array} \quad (7.21)$$

Moreover, in the diagram below, the right vertical arrow is an equivalence when

$i > 0$.

$$\begin{array}{ccc} \mathcal{B}_i & \longrightarrow & \mathcal{B}_i/\mathcal{B}_{i-1} \\ \downarrow & & \downarrow \simeq \\ \mathcal{C}^\oplus & \longrightarrow & \mathcal{C}^\oplus/\text{add}(\bigoplus_{l \neq i} M_l) \end{array}$$

Therefore for $i > 0$, the map $K_1(\mathcal{B}_i) \rightarrow K_1(\mathcal{B}_i/\mathcal{B}_{i-1})$ factors through F_i as indicated in (7.21). It follows that the right vertical map in (7.21) is an isomorphism.

So there are short exact sequences

$$0 \longrightarrow F_{i-1} \longrightarrow F_i \longrightarrow K_1(\mathcal{B}_i/\mathcal{B}_{i-1}) \longrightarrow 0 \quad (7.22)$$

For each i , $\mathcal{B}_i/\mathcal{B}_{i-1}$ has one nonzero indecomposable M_i . The ring homomorphism $k[[t^2, t^{2(n-i)+1}]] \rightarrow \text{End}_R(M_i)$, sending f to the multiplication-by- f endomorphism, is an isomorphism for each i . Using this and the AR quiver (7.20), we see that for $i > 0$,

$$\text{End}_{\mathcal{B}_i/\mathcal{B}_{i-1}} M_i = (\text{End}_R M_i)/(t^2) = \begin{cases} k & \text{if } 0 < i < n \\ k[t]/(t^2) & \text{if } i = n \end{cases}$$

so that $\mathcal{B}_i/\mathcal{B}_{i-1} \simeq \text{mod}(k)$ if $0 < i < n$, and $\mathcal{B}_n/\mathcal{B}_{n-1} \simeq \text{proj}(k[t]/(t^2))$. Let k^+ be the additive abelian group of k . Then $(k[t]/(t^2))^\times \cong k^\times \oplus k^+$ via the identification $\alpha(1 + \beta t) \mapsto (\alpha, \beta)$. Therefore

$$K_1(\mathcal{B}_i/\mathcal{B}_{i-1}) = K_1(\text{End}_{\mathcal{B}_i/\mathcal{B}_{i-1}} M_i) = \begin{cases} k^\times & \text{if } 0 < i < n \\ k^\times \oplus k^+ & \text{if } i = n \end{cases}$$

So, according to the sequences (7.22) and the descriptions above of the groups $K_1(\mathcal{B}_i/\mathcal{B}_{i-1})$, there is a filtration

$$0 \subset F_0 \subset \cdots \subset F_n = K_1(\mathcal{C}^\oplus)$$

such that

1. F_0 is a quotient of $K_1(R) = R^\times$.
2. $F_i/F_{i-1} \cong k^\times$ for $i = 1, \dots, n-1$.
3. $F_n/F_{n-1} \cong k^\times \oplus k^+$.

The group k^\times appears in this filtration as a subquotient of $K_1(\mathcal{C}^\oplus)$ $n+1$ times: it appears as a subobject of F_0 (we shall soon see that the composition $k^\times \rightarrow R^\times \rightarrow F_0$ is monic); it appears $n-1$ times as F_i/F_{i-1} , $0 < i < n$; and it appears as a summand of F_n/F_{n-1} . We next argue that each of these copies of k^\times is in fact a summand of $K_1(\mathcal{C}^\oplus)$.

Let

$$m_i : \text{mod}(k) \rightarrow \mathcal{C}^\oplus \quad \text{and} \quad j_i : \text{mod}(k) \rightarrow \mathcal{B}_i$$

be the k -linear functors which (both) send k to M_i , and let

$$m = \bigoplus m_i : (\text{mod}(k))^{\oplus n+1} \rightarrow \mathcal{C}^\oplus \quad \text{and} \quad m' = \bigoplus_{i>0} m_i : (\text{mod}(k))^{\oplus n} \rightarrow \mathcal{C}^\oplus.$$

Let $q : \mathcal{C}^\oplus \rightarrow \mathcal{C}^\oplus/\text{rad}_{\mathcal{C}^\oplus}$ be the quotient functor and

$$\text{proj}_i : \mathcal{C}^\oplus/\text{rad}_{\mathcal{C}^\oplus} \simeq (\text{mod}(k))^{\oplus n+1} \rightarrow \text{mod}(k)$$

the i th projection. Then the diagram below commutes.

$$\begin{array}{ccc}
 \text{mod}(k) & \xrightarrow{=} & \text{mod}(k) \\
 \downarrow j_i & & \uparrow \\
 \mathcal{B}_i & \xrightarrow{p_i} & \mathcal{B}_i/\mathcal{B}_{i-1} \\
 \downarrow & & \downarrow \\
 \mathcal{C}^\oplus & \xrightarrow{q} & \mathcal{C}^\oplus/\text{rad}_{\mathcal{C}^\oplus}
 \end{array}
 \begin{array}{l}
 \\
 \\
 \\
 \text{proj}_i
 \end{array}$$

From this we deduce the following.

1. The map $k^\times \rightarrow F_0$ induced by j_0 has a left inverse which factors through $K_1(\mathcal{C}^\oplus)$. Therefore k^\times is embedded in F_0 in such a way that is a summand of $K_1(\mathcal{C}^\oplus)$.

2. For $0 < i < n$, $K_1(j_i)$ embeds k^\times as a summand of $K_1(\mathcal{C}^\oplus)$ which is contained in F_i and projects isomorphically onto F_i/F_{i-1} .
3. $K_1(j_n)$ embeds k^\times as a summand of $K_1(\mathcal{C}^\oplus)$, and $K_1(p_n j_n) : k^\times \rightarrow k^\times \oplus k^+$ is the inclusion into the first coordinate.

We compile all of this data in the following commuting diagram, in which all rows are split short exact sequences and all columns are exact.

$$\begin{array}{ccccccc}
0 & \longrightarrow & k^\times & \longrightarrow & R^\times & \longrightarrow & R^\times/k^\times \longrightarrow 0 \\
& & \downarrow & \searrow^{K_1(m_0)} & \downarrow^{K_1(r)} & & \downarrow \\
0 & \longrightarrow & (k^\times)^{n+1} & \xrightarrow{K_1(m)} & K_1(\mathcal{C}^\oplus) & \longrightarrow & \text{coker } K_1(m) \longrightarrow 0 \\
& & \downarrow & & \downarrow^{\pi_1(\rho)} & & \downarrow \\
& & (k^\times)^n & & K_1(X) & & k^+ \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array} \tag{7.23}$$

Note that $\pi_1(\rho)$ is surjective because there is an exact sequence

$$\begin{array}{ccccccc}
K_1(\mathcal{C}^\oplus) & \xrightarrow{\pi_1(\rho)} & K_1(X) & \longrightarrow & K_0(R) & \longrightarrow & K_0(\mathcal{C}^\oplus) \\
& & & & \parallel & & \parallel \\
& & & & \mathbb{Z} & \longrightarrow & \mathbb{Z}^{n+1}
\end{array}$$

Therefore the terms in the third row of (7.23) fit into an exact sequence

$$(k^\times)^n \xrightarrow{\pi_1(\rho) \circ K_1(m')} K_1(X) \longrightarrow k^+ \longrightarrow 0.$$

We next argue that the first map in this sequence is injective. For this it suffices to show that $\text{im } K_1(m) \cap \text{im } K_1(r) \subset \text{im } K_1(m_0)$. For $i \neq 0$, the composition $\text{proj}(R) \xrightarrow{r} \mathcal{C}^\oplus \xrightarrow{\text{proj}_i \circ q} \text{mod}(k)$ is zero, so $\text{im } K_1(r) \subset \ker K_1(\text{proj}_i \circ q)$ and there-

fore

$$\begin{aligned}
\operatorname{im} K_1(m) \cap \operatorname{im} K_1(r) &\subset \operatorname{im} K_1(m) \cap \left(\bigcap_{i>0} \ker K_1(\operatorname{proj}_i \circ q) \right) \\
&= K_1(m) \left(\bigcap_{i>0} \ker K_1(\operatorname{proj}_i \circ q \circ m) \right) \\
&= \operatorname{im} K_1(m_0),
\end{aligned}$$

as desired.

Now using (7.5) and Proposition 7.14 one obtains an exact sequence

$$(k^\times)^n \xrightarrow{K_1(m) \circ (T \cdot \operatorname{id}_{k^\times})} K_1(\mathcal{C}^\oplus) \longrightarrow K'_1(R) \longrightarrow (K_0(k))^n \longrightarrow K_0(\mathcal{C}^\oplus) \quad (7.24)$$

$$\begin{array}{ccc}
& \parallel & \parallel \\
\mathbb{Z}^n & \xrightarrow{T} & \mathbb{Z}^{n+1}
\end{array}$$

Similarly, by (7.10) and Proposition 7.19 there is an exact sequence

$$(k^\times)^n \xrightarrow{\pi_1(\rho) \circ K_1(m') \circ (T' \cdot \operatorname{id}_{k^\times})} K_1(X) \longrightarrow K_1(\mathbf{MF}) \longrightarrow (K_0(k))^n \longrightarrow K_0(X) \quad (7.25)$$

$$\begin{array}{ccc}
& \parallel & \parallel \\
\mathbb{Z}^n & \xrightarrow{T'} & \mathbb{Z}^n
\end{array}$$

The matrices T and T' can be computed directly from the Auslander-Reiten quiver (7.20); keeping in mind our convention that T has a zeroth row but no zeroth column, these matrices are

$$T = \begin{pmatrix} -1 & 0 & 0 & & & & & & \\ & 2 & -1 & 0 & & & & & \\ & -1 & 2 & -1 & \cdots & & & & \\ & 0 & -1 & 2 & & & 0 & & \\ & 0 & 0 & -1 & & & -1 & 0 & \\ & 0 & 0 & 0 & \ddots & & 2 & -1 & \\ & & \vdots & & & & -1 & 1 & \end{pmatrix}; \text{ explicitly, } T_{lj} = \begin{cases} -1 & \text{if } j = l \pm 1 \\ 2 & \text{if } j = l < n \\ 1 & \text{if } j = l = n \\ 0 & \text{otherwise} \end{cases}$$

$$T' = \begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & \cdots & & \\ 0 & -1 & 2 & & 0 & \\ 0 & 0 & -1 & & -1 & 0 \\ 0 & 0 & 0 & \ddots & 2 & -1 \\ & \vdots & & & -1 & 1 \end{pmatrix}.$$

One proves easily by induction that $\det T' > 0$, so the last map in each sequence (7.24) and (7.25) is injective. Since these sequences are exact, it follows that

$$\begin{aligned} K_1'(R) &\cong \operatorname{coker}[K_1(m) \circ (T \cdot \operatorname{id}_{k^\times})] \\ K_1(\mathbf{MF}) &\cong \operatorname{coker}[\pi_1(\rho) \circ K_1(m') \circ (T' \cdot \operatorname{id}_{k^\times})] \end{aligned}$$

Together with the data from (7.23), we obtain the following decompositions.

Proposition 7.26.

1. *There is an abelian group G ($= \operatorname{coker} K_1(m)$) such that*

$$K_1(\mathcal{C}^\oplus) \cong \operatorname{coker}(T \cdot \operatorname{id}_{k^\times}) \oplus G$$

and G fits into an exact sequence

$$R^\times/k^\times \longrightarrow G \longrightarrow k^+ \longrightarrow 0.$$

2. *There is a short exact sequence*

$$0 \longrightarrow \operatorname{coker}(T' \cdot \operatorname{id}_{k^\times}) \longrightarrow K_1(\mathbf{MF}) \longrightarrow k^+ \longrightarrow 0.$$

7.4 Dimension Two

The following theorem is due to Henrik Holm and Lars Winther Christensen.

Proposition 7.27 ([Hol12, Lemma 2.4]). *Suppose R is an ADE singularity (see Definition 3.25) and $\dim(R)$ is even. Then the stable Auslander-Reiten matrix T' is injective.*

Proof. When R is an ADE singularity of even dimension, the Auslander-Reiten quiver of R is the double of a simply laced Dynkin graph G , and for any nonfree indecomposable M , $\tau M = M$ (see [Aus87][Theorem 1]). This means that the Auslander-Reiten sequence ending in a nonfree indecomposable M_i is of the form

$$0 \longrightarrow M_i \longrightarrow \bigoplus M_j \longrightarrow \bigoplus M_i \longrightarrow 0$$

where the direct sum is over all j such that nodes j and i are adjacent in G . Therefore the Auslander-Reiten matrix T has the form

$$T_{ij} = \begin{cases} -1 & \text{if } j \neq i \text{ and } j \text{ and } i \text{ are adjacent} \\ 2 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

This is exactly the Cartan matrix associated to the Dynkin graph G . It is known to be nonsingular. □

7.5 Dimension Three

The following argument is due to Michael Wemyss. Let R be a Gorenstein complete local ring of dimension 3 over an algebraically closed field, and let $L \in \text{CM}(R)$ be a 2-cluster tilting object. Adopt the notation of 7.2: $\Lambda := \text{End}_R(L)^{\text{op}}$; M^0, \dots, M^t are the indecomposable summands of L ; and S^0, \dots, S^t are the corresponding simple Λ -modules. By Theorem 6.15, Λ is 3-Calabi-Yau, so there are natural isomorphisms

$$\text{Ext}_\Lambda^i(S^j, S^l) \cong \text{Ext}_\Lambda^{3-i}(S^l, S^j),$$

This allows us to simplify the Auslander-Reiten matrix considerably:

$$\begin{aligned}
T_{lj} &= -\dim \operatorname{Ext}_\Lambda^3(S^j, S^l) + \dim \operatorname{Ext}_\Lambda^2(S^j, S^l) \\
&\quad - \dim \operatorname{Ext}_\Lambda^1(S^j, S^l) + \dim \operatorname{Ext}_\Lambda^0(S^j, S^l) \\
&= -\dim \operatorname{Hom}_\Lambda(S^l, S^j) + \dim \operatorname{Ext}_\Lambda^1(S^l, S^j) \\
&\quad - \dim \operatorname{Ext}_\Lambda^1(S^j, S^l) + \dim \operatorname{Hom}_\Lambda(S^j, S^l) \\
&= \dim \operatorname{Ext}_\Lambda^1(S^l, S^j) - \dim \operatorname{Ext}_\Lambda^1(S^j, S^l)
\end{aligned}$$

Now, $\dim \operatorname{Ext}_\Lambda^1(S^j, S^l)$ is the number of arrows from node j to node l in the quiver of Λ . This quiver is known to be symmetric, so $\dim \operatorname{Ext}_\Lambda^1(S^j, S^l) = \dim \operatorname{Ext}_\Lambda^1(S^l, S^j)$. This proves the following.

Proposition 7.28. *Suppose R is Gorenstein of dimension 3 and $\operatorname{CM}(R)$ has a 2-cluster tilting object L . Then the Auslander-Reiten matrix T is zero. In particular, $K'_0(R)$ is the free abelian group generated by the classes of indecomposable summands of L .*

8 Noncommutative Localizations

The goal of this section is to explain what we know about the possibility to extend the sequence (5.14) to the left when Λ is the endomorphism ring of an n -cluster tilting object of $\operatorname{CM}(R)$. Sections 8.1 and 8.2 are background, and section 8.3 contains the main results.

8.1 Noncommutative Localizations and Homological Ring Epimorphisms

Definition 8.1. A ring homomorphism $\phi : R \rightarrow S$ is called a *ring epimorphism* if it is an epimorphism in the category of rings.

Remark 8.2. Of course, a surjective ring homomorphism is always a ring epimorphism. But the converse is emphatically false – any localization of a commutative ring, for example, is a ring epimorphism.

Proposition 8.3. *Let $\phi : R \rightarrow S$ be a ring homomorphism. The following are equivalent:*

1. ϕ is a ring epimorphism.
2. The restriction $\phi_* : \mathbf{Mod}(S) \rightarrow \mathbf{Mod}(R)$ is fully faithful.
3. $\phi \otimes_R S = S \otimes_R \phi : S \rightarrow S \otimes_R S$ is an isomorphism of S - S -bimodules.

Proposition 8.4 ([Sch85, Theorem 4.1]). *Let R be a ring and Σ a set of maps between finitely generated projective left R -modules. There is a ring homomorphism $l_\Sigma : R \rightarrow R_\Sigma$, called the universal localization at Σ , satisfying the following two properties.*

1. For any $\sigma \in \Sigma$, $R_\Sigma \otimes_R \sigma$ is an isomorphism of left R -modules.
2. Any ring homomorphism $R \rightarrow S$ such that $S \otimes_R \sigma$ is invertible for all $\sigma \in \Sigma$ factors uniquely through l_Σ .

$$\begin{array}{ccc}
 R & \xrightarrow{\quad} & S \\
 & \searrow & \nearrow \\
 & l_\Sigma & R_\Sigma
 \end{array}$$

For the rest of this section, let Σ be a set of maps between finitely generated projective R -modules, and let $l_\Sigma : R \rightarrow R_\Sigma$ be the universal localization at Σ .

Proposition 8.5. *l_Σ is a ring epimorphism, and*

$$\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0.$$

Proof. Let

$$0 \longrightarrow M \xrightarrow{f} F \xrightarrow{g} R_\Sigma \longrightarrow 0$$

be an exact sequence of left R -modules with F a free R -module. Consider the exact sequence of left R_Σ -modules

$$0 \longrightarrow \mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) \longrightarrow R_\Sigma \otimes_R M \longrightarrow R_\Sigma \otimes_R F \longrightarrow R_\Sigma \otimes_R R_\Sigma = R_\Sigma \longrightarrow 0$$

$\searrow \quad \nearrow$
 Z

For any left R_Σ -module N , $\mathrm{Ext}_R^1(R_\Sigma, N) = \mathrm{Ext}_{R_\Sigma}^1(R_\Sigma, N) = 0$, so the horizontal maps below are surjective.

$$\begin{array}{ccc}
 \mathrm{Hom}_R(F, N) & \xrightarrow{\hspace{10em}} & \mathrm{Hom}_R(M, N) \\
 \parallel & & \parallel \\
 \mathrm{Hom}_{R_\Sigma}(R_\Sigma \otimes_R F, N) & \xrightarrow{\mathrm{Hom}_{R_\Sigma}(R_\Sigma \otimes_R f, N)} & \mathrm{Hom}_{R_\Sigma}(R_\Sigma \otimes_R M, N) \\
 \searrow & & \nearrow \\
 & \mathrm{Hom}_{R_\Sigma}(Z, N) & \\
 & \xi_N &
 \end{array}$$

Therefore the map labeled ξ_N is also surjective. ξ_N also fits into the following exact sequence.

$$\begin{array}{ccc}
 0 & \longrightarrow & \mathrm{Hom}_{R_\Sigma}(Z, N) \xrightarrow{\xi_N} \mathrm{Hom}_{R_\Sigma}(R_\Sigma \otimes_R M, N) \\
 & & \searrow \\
 & & \mathrm{Hom}_{R_\Sigma}(\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma), N) \longrightarrow \mathrm{Ext}_{R_\Sigma}^1(Z, N)
 \end{array}$$

Therefore $\mathrm{Hom}_{R_\Sigma}(\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma), N)$ is a submodule of $\mathrm{Ext}_{R_\Sigma}^1(Z, N)$. But $\mathrm{Ext}_{R_\Sigma}^1(Z, N)$ fits into the exact sequence

$$\begin{array}{ccccc}
 \mathrm{Ext}_{R_\Sigma}^2(R_\Sigma \otimes_R F, N) & \longrightarrow & \mathrm{Ext}_{R_\Sigma}^1(Z, N) & \longrightarrow & \mathrm{Ext}_{R_\Sigma}^1(R_\Sigma, N) \\
 \underbrace{\quad}_0 & & & & \underbrace{\quad}_0
 \end{array}$$

so $\mathrm{Ext}_{R_\Sigma}^1(Z, N) = 0$ and therefore $\mathrm{Hom}_{R_\Sigma}(\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma), N) = 0$. Since N was arbitrary, $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$. \square

Definition 8.6. A ring epimorphism $\phi : R \rightarrow S$ is called *homological* if, for all $i > 0$, $\mathrm{Tor}_i^R(S, S) = 0$.

Remark 8.7. Not every universal localization is a homological ring epimorphism, as an example of Marks-Vitória shows ([MV12, Example 2.14]). Not every homological ring epimorphism is a universal localization, as an example of Keller shows ([Kel94]).

Proposition 8.8 ([MV12, Theorem 3.3]). *Suppose $\phi : R \rightarrow S$ is a ring epimorphism that makes S into a finitely presented R -module of projective dimension at most 1. Then ϕ is a homological epimorphism if and only if it is a universal localization.*

Definition 8.9. An idempotent two-sided ideal I of R is called *stratifying* if the following conditions hold.

1. $I = ReR$ for some idempotent $e \in R$.
2. The multiplication map $Re \otimes_{eRe} eR \rightarrow I$ is an isomorphism.
3. $\mathrm{Tor}_i^{eRe}(Re, eR) = 0$ for $i > 0$.

Proposition 8.10. *If ReR is stratifying, then the quotient $p : R \rightarrow R/ReR$ is a homological ring epimorphism.*

Proof. Suppose ReR is stratifying, so there is a quasi-isomorphism

$$Re \otimes_{eRe}^{\mathbb{L}} eR \xrightarrow{\simeq} ReR.$$

Therefore there is a quasi-isomorphism

$$(Re \otimes_{eRe}^{\mathbb{L}} eR) \otimes_R^{\mathbb{L}} (Re \otimes_{eRe}^{\mathbb{L}} eR) \xrightarrow{\simeq} ReR \otimes_R^{\mathbb{L}} ReR.$$

Now, since Re is a projective left R -module, $eR \otimes_R^{\mathbb{L}} Re = eR \otimes_R Re$, so

$$\begin{aligned}
(Re \otimes_{eRe}^{\mathbb{L}} eR) \otimes_R^{\mathbb{L}} (Re \otimes_{eRe}^{\mathbb{L}} eR) &\simeq Re \otimes_{eRe}^{\mathbb{L}} (eR \otimes_R^{\mathbb{L}} Re) \otimes_{eRe}^{\mathbb{L}} eR \\
&\simeq Re \otimes_{eRe}^{\mathbb{L}} (eR \otimes_R Re) \otimes_{eRe}^{\mathbb{L}} eR \\
&\simeq Re \otimes_{eRe}^{\mathbb{L}} eRe \otimes_{eRe}^{\mathbb{L}} eR \\
&\simeq Re \otimes_{eRe} eRe \otimes_{eRe} eR \\
&\simeq ReR
\end{aligned}$$

Therefore $ReR \otimes_R^{\mathbb{L}} ReR$ is acyclic in nonzero degrees. From the long exact sequence

$$\cdots \rightarrow \mathrm{Tor}_i^R(ReR, ReR) \rightarrow \mathrm{Tor}_i^R(ReR, R) \rightarrow \mathrm{Tor}_i^R(ReR, R/ReR) \rightarrow \cdots$$

and the fact that the map $ReR \otimes_R ReR \rightarrow ReR \otimes_R R$ is an isomorphism, we see that $\mathrm{Tor}_i^R(ReR, R/ReR) = 0$ for $i > 0$. From the long exact sequence

$$\cdots \rightarrow \mathrm{Tor}_i^R(ReR, R/ReR) \rightarrow \mathrm{Tor}_i^R(R, R/ReR) \rightarrow \mathrm{Tor}_i^R(R/ReR, R/ReR) \rightarrow \cdots$$

and the fact that $ReR \otimes_R R/ReR = 0$, we see that $\mathrm{Tor}_i^R(R/ReR, R/ReR) = 0$ for $i > 0$. This proves $p : R \rightarrow R/ReR$ is homological. \square

8.2 Noncommutative Localizations in K -Theory

Definition 8.11. Let R be a ring and Σ a set of maps between finitely generated projective left R -modules. Let $\mathrm{D}^{\mathrm{perf}}(R)$ be the perfect derived category of R , i.e. the homotopy category of $\mathrm{Ch}^{\mathrm{b}}(\mathrm{proj}(R))$. Each $\sigma : P \rightarrow Q$ in Σ defines a complex

$$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{\sigma} Q \longrightarrow 0 \longrightarrow \cdots$$

with, say, Q in degree 0. Let $\mathrm{D}(R, \Sigma)$ be the thick triangulated subcategory of $\mathrm{D}^{\mathrm{perf}}(R)$ generated by the maps in Σ , and let $\mathrm{Ch}^{\mathrm{b}}(R, \Sigma)$ be the full subcategory

of $\text{Ch}^b(\text{proj}(R))$ consisting of complexes isomorphic in $\text{D}^{\text{perf}}(R)$ to a complex in $\text{D}(R, \Sigma)$.

Let $\mathcal{T} := (\text{D}^{\text{perf}}(R)/\text{D}(R, \Sigma))$ be the Verdier quotient of $\text{D}^{\text{perf}}(R)$ by $\text{D}(R, \Sigma)$. Since any complex in Σ is contractible as a complex of R_Σ -modules, the composition of horizontal functors below is zero.

$$\begin{array}{ccccc} \text{D}(R, \Sigma) & \longrightarrow & \text{D}^{\text{perf}}(R) & \xrightarrow{l_\Sigma^*} & \text{D}^{\text{perf}}(R_\Sigma) \\ & & & \searrow & \nearrow i \\ & & & \mathcal{T}^c & \end{array}$$

Since, in addition, $\text{D}^{\text{perf}}(R_\Sigma)$ is idempotent complete, there is a factorization i , as indicated, of l_Σ^* through the idempotent completion \mathcal{T}^c of \mathcal{T} .

We say the pair (R, Σ) *satisfies Waldhausen localization* if i is an equivalence. If this is true, it follows from 4.14 and 4.13.1 that $K(\text{Ch}^b(R, \Sigma))$ is the (-1)-connected cover of the homotopy fiber of $K(R) \rightarrow K(R_\Sigma)$, so that

$$K(\text{Ch}^b(R, \Sigma)) \longrightarrow K(R) \xrightarrow{l_\Sigma^*} K(R_\Sigma)$$

induces a long exact sequence of homotopy groups, ending in

$$\dots \longrightarrow K_0(\text{Ch}^b(R, \Sigma)) \longrightarrow K_0(R) \longrightarrow K_0(R_\Sigma) .$$

Theorem 8.12 ([TT90]). *Let R be a commutative ring and $\Sigma \subset R$ a multiplicative set, viewed as a set of maps $R \rightarrow R$. Then (R, Σ) satisfies Waldhausen localization.*

Theorem 8.13 ([WY92]). *Let R be a ring and $\Sigma \subset R$ a multiplicative set satisfying the following conditions:*

1. *the Ore condition: for any $\sigma \in \Sigma$, $r \in R$, there are $\sigma' \in \Sigma$ and $r' \in R$ such that $\sigma r = r' \sigma'$.*
2. *for any $\sigma \in \Sigma$ and $r \in R$ such that $\sigma r = 0$, there is $\tau \in \Sigma$ such that $\tau \sigma = 0$.*

Then (R, Σ) satisfies Waldhausen localization.

Theorem 8.14 ([NR04]). *Suppose l_Σ is a homological ring epimorphism. Then (R, Σ) satisfies Waldhausen localization.*

Remark 8.15. In the next section we shall see how to produce a pair (R, Σ) which does not satisfy the hypotheses of either 8.13 or 8.14, but nevertheless has the property that

$$K_1(R, \Sigma) \longrightarrow K_1(R) \longrightarrow K_1(R_\Sigma) \xrightarrow{0} K_0(R, \Sigma) \longrightarrow K_0(R) \longrightarrow K_0(R_\Sigma) \longrightarrow 0$$

is exact.

8.3 These ring epimorphisms are not homological

In this section, let R be a complete local Gorenstein ring of Krull dimension d , and suppose R has a $(d-1)$ -cluster tilting object L . Recall from 7.4 the homotopy fiber sequence

$$\bigvee_{[M] \in \text{ind}_0(\mathcal{C})} K(\kappa_M) \longrightarrow K'(\Lambda) \longrightarrow K'(R) \tag{8.16}$$

We have seen in 7.3 that using this sequence, information about $K'(\Lambda)$ yields information about $K'(R)$. In Section 5 we deduced a way to decompose $K_1(\Lambda)$, and we demonstrated in 7.3 how this decomposition could be used to describe $K'_1(R)$. If one could extend the sequence (5.14), one could describe $K'_i(R)$ in similar fashion. Theorem 8.14 gives hope that it is possible to extend the sequence. In this section we give evidence that this is impossible, as the hypotheses of the Neeman-Ranicki theorem are often not satisfied in our setting. The argument is due to Michael Wemyss.

We first need a bit of background. We adopt our earlier notation: $\Lambda = \text{End}_R(L)^{\text{op}}$, $e \in \Lambda$ a nonzero idempotent corresponding to a summand $N = \text{im}(e)$

of L . We assume N is not an additive generator of $\mathcal{C} = \text{add}(L)$, and we need the additional assumption that $e\Lambda e$ is a maximal Cohen-Macaulay R -module. This last assumption is equivalent to $\text{Ext}_R^i(X, X) = 0$ for $0 < i < d - 1$ ([Iya07b, 2.5.1]). It is not always satisfied, but it often is; see [Leu12, §M].

Lemma 8.17. *$\Lambda/\Lambda e\Lambda$ is a finite length R -module.*

Proof. It suffices to show that $(\Lambda/\Lambda e\Lambda)_{\mathfrak{p}} = 0$ for any nonmaximal prime ideal \mathfrak{p} of R . Let $L = \bigoplus M_i$ be the decomposition of L into indecomposables, and let \mathfrak{p} be a nonmaximal prime of R . Since each M_i is maximal Cohen-Macaulay, $(M_i)_{\mathfrak{p}}$ is a maximal Cohen-Macaulay module over the regular local ring $R_{\mathfrak{p}}$, so $(M_i)_{\mathfrak{p}} \cong R_{\mathfrak{p}}$. Then

$$\Lambda_{\mathfrak{p}} = (\text{End}_R(L))_{\mathfrak{p}} = \left(\text{End}_R \left(\bigoplus M_i \right) \right)_{\mathfrak{p}} = \text{End}_{R_{\mathfrak{p}}} \left(\bigoplus (M_i)_{\mathfrak{p}} \right) \cong \text{End}_{R_{\mathfrak{p}}} \left(\bigoplus R_{\mathfrak{p}} \right)$$

Since $R_{\mathfrak{p}}$ is a local ring, any idempotent of $\text{End}_{R_{\mathfrak{p}}}(\bigoplus R_{\mathfrak{p}})$ generates the whole ring, so $(\Lambda e\Lambda)_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} e \Lambda_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}$ and therefore $(\Lambda/\Lambda e\Lambda)_{\mathfrak{p}} = 0$. \square

We will also need the following facts; proofs are omitted. The first is a generalization of 3.17 for orders.

Theorem 8.18 ([Iya08]). *Let Λ be an R -order which is an isolated singularity. Then $\underline{\text{Hom}}(\tau Y, X) \cong D\text{Ext}_{\Lambda}^1(X, Y)$.*

Proposition 8.19 ([Iya08, Proposition 3.22]). *Let Λ be a symmetric R -order which is an isolated singularity. Then there is an isomorphism $\tau \cong \Omega^{2-d}$ of functors $\underline{\text{CM}}(\Lambda) \rightarrow \underline{\text{CM}}(\Lambda)$.*

Proposition 8.20 (M. Wemyss). *$\Lambda e\Lambda$ is not stratifying.*

Proof. Suppose $\Lambda e\Lambda$ is stratifying, so $\Lambda e \otimes_{e\Lambda e}^{\mathbb{L}} e\Lambda \simeq \Lambda$. Then

$$\begin{aligned}
\Lambda &\simeq \mathbb{R}\mathrm{Hom}_{\Lambda}(\Lambda, \Lambda) \\
&\simeq \mathbb{R}\mathrm{Hom}_{\Lambda}(\Lambda e \otimes_{e\Lambda e}^{\mathbb{L}} e\Lambda, \Lambda) && \text{as } \Lambda \text{ is stratifying} \\
&\simeq \mathbb{R}\mathrm{Hom}_{e\Lambda e}(e\Lambda, \mathbb{R}\mathrm{Hom}_{\Lambda}(\Lambda e, \Lambda)) \\
&\simeq \mathbb{R}\mathrm{Hom}_{e\Lambda e}(e\Lambda, e\Lambda) && \text{as } \Lambda e \in \mathbf{proj}(\Lambda)
\end{aligned}$$

In particular $\mathrm{Ext}_{e\Lambda e}^i(e\Lambda, e\Lambda) = 0$ for $i > 0$.

On the other hand, $e\Lambda e$ is the endomorphism ring of the summand $\mathrm{im}(e)$ of L , so by 6.16 is d -CY⁻. By Auslander-Reiten duality 8.18 and Proposition 8.19,

$$\begin{aligned}
\underline{\mathrm{Hom}}_{e\Lambda e}(e\Lambda, e\Lambda) &= D\mathrm{Ext}_{e\Lambda e}^1(e\Lambda, \tau e\Lambda) \\
&= D\mathrm{Ext}_{e\Lambda e}^{d-1}(e\Lambda, e\Lambda) \\
&= 0
\end{aligned}$$

Therefore $e\Lambda$ is a projective $e\Lambda e$ module, so $M \in \mathbf{add}(N)$ and N is an additive generator of \mathcal{C} , contradicting our hypothesis on N . \square

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